

# Pricing Algorithms of Multivariate Path Dependent Options

Yue Kuen Kwok\*

*Department of Mathematics, Hong Kong University of Science and Technology,  
Clear Water Bay, Hong Kong*

Hoi Ying Wong

*Department of Mathematics, Hong Kong University of Science and Technology,  
Clear Water Bay, Hong Kong*

and

Ka Wo Lau

*Department of Computer Science, Hong Kong University of Science and  
Technology, Clear Water Bay, Hong Kong*

Received 10 January 2000; revised 25 June 2000

Financial derivatives which are multivariate in nature are abundant in the financial markets. The underlying state variables may be the stock prices, interest rates, exchange rates, stochastic volatility, average of stock prices, extremum values of stock prices, etc. Option contracts whose life and payoff depending on the stochastic movement of the underlying asset prices are termed path dependent options. In this paper, we examine the pricing methods of several prototype path dependent options. These include options with sequential barriers, options with an external barrier and two-asset lookback options. The governing equations for the option prices are seen to resemble the diffusion type equations but with cross derivative terms, a feature which differs from the usual diffusion equations in engineering. Various techniques to reduce the complexity of the multi-variate nature of these prototype option pricing models are discussed. It is illustrated that the dimensionality of a path dependent option model may be reduced by some ingenious choices of similarity variables. We also examine the design of pricing algorithms of these multi-variate options, in particular, with regard to the treatment of discrete monitoring feature and the prescription of numerical boundary conditions. The possible generalizations of the numerical techniques presented in this paper to other models with more complicated path dependent payoff structures are also discussed.

\* Research supported by the Research Grants Council of Hong Kong, HKUST 6250/97H

## 1. INTRODUCTION

By applying the riskless hedging principle, Black and Scholes [1] pioneered the development of the formulation of the pricing models of financial derivatives. Since then, the complexity in the design of financial derivatives has grown tremendously over the past decades. This is attributed to the competition pressure among financial institutions for more innovative risk management tools that are tailored to the specific needs of customers. In particular, there has been a growing popularity for path dependent options, so named since the payouts and the stopping times of these options depend on the stochastic movement of the underlying asset prices during the whole or part of the life of the options. The common path dependent features are the barrier feature, Asian feature and lookback feature.

The multi-variate nature of the option pricing models arises from the multiplicity of risk factors in the financial derivatives. The risk factors include the prices of the underlying assets, interest rates, exchange rates, stochastic volatility, average of asset prices, extremum values of asset prices, etc. The discrete monitoring of the path dependent feature may also lead to multi-variate nature, which arises from the correlation of the Geometric Brownian magnifications of the asset price ratios corresponding to overlapping time intervals.

In this paper, we would like to develop pricing methodologies for multi-variate path dependent options, and illustrate the successes and limitations of these techniques through the solution of several prototype option models. The complexity of the pricing methods grows with the dimensionality of the models. Since the governing equations of the option models are parabolic in nature, one may achieve the reduction of dimensionality by some ingenious choices of similarity variables. Also, though the analytical price formulas of discretely monitored path dependent options normally involve  $n$ -dimensional cumulative distribution functions, where  $n$  is the number of monitoring instants, a carefully designed valuation algorithms may reduce the pricing problem into a succession of one-dimensional problems.

This paper is organized as follows. In the next section, we develop the analytical price formulas and numerical algorithms for the pricing of options with sequential barriers. The barriers can be monitored discretely or continuously. In Section III, we construct the finite difference scheme for two-asset option models, in particular, we address the tricky issue of the prescription of numerical boundary conditions. Sample calculations are performed to price options with single external barrier and two-asset lookback options. The advantages and disadvantages of constructing the difference schemes along the computational boundaries using the technique of skew computational stencils are carefully examined. The paper is ended with summaries and conclusions in the last section.

## 2. OPTIONS WITH SEQUENTIAL BARRIERS

A new class of barrier contracts having two barriers but with sequential breaching requirement have been structured in the financial markets. Unlike the usual two-sided barrier options, the order of breaching of the barriers is specified. The second barrier is activated only after the first barrier has been hit earlier, and the option is knocked out only if both barriers have been breached in the pre-specified order. The added feature of choosing the order of breaching and positions of the barriers gives the investors more flexibility to design the desired barrier clauses that fit their views on the movement of the asset price.

By adopting the usual Black-Scholes pricing framework, the pricing of continuously monitored European sequential barrier options has been recently considered by Li [10] and Sidenius [13]. Here, we develop the explicit analytic representation of the price formulas and valuation algorithms for both continuously and discretely monitored European sequential barrier options. Note that if the first barrier has been hit earlier, then the option reduces to the usual single barrier option. Hence, it is only necessary to consider the situation where the first barrier has never been hit.

### 2.1. Continuously monitored sequential barriers

The continuously monitored barrier models may not quite reflect market reality since the continuous monitoring of the asset price movement on the breaching of barrier is almost prohibitively impossible in real market situations. Rather, daily or weekly discrete monitoring is usually taken in barrier option contracts. However, the discussion of the continuously monitored case is included here since it exhibits elegant analytical tractability and corresponds to the limiting case of infinite number of discretely monitored instants.

Let the first barrier be an upstream barrier  $B_H$  and the second barrier be a downstream barrier  $B_L$  ( $B_L < B_H$ ), both barrier levels are taken to be constant. We would like to derive the pricing formula of this European sequential barrier call option with strike price  $X$  and on an underlying asset of price  $S$ . Since the second barrier  $B_L$  is activated only when the first barrier  $B_H$  is hit, the sequential barrier call option behaves like a European up-and-out call with barrier  $B_H$  and the rebate at the asset price  $S = B_H$  is a down-and-out call with barrier  $B_L$ . This is because when the first barrier is hit, the sequential barrier option can be essentially replaced by a down-and-out call as the rebate.

Let  $c_{up-out}(S, \tau; B_H)$  denote the option value of a European up-and-out call with barrier  $B_H$  and  $c_{down-out}(S, \tau; B_L)$  denote the down-and-out counterpart with barrier  $B_L$ , where  $\tau$  is the time to expiry. Also, let  $Q(\omega; S, B_H)$  denote the density function of the first passage time when the first barrier

$B_H$  is hit, where  $\omega$  is the time lapsed from the current time. The price of the present call with sequential barriers is given by

$$c_{seq}(S, \tau; B_H, B_L) = c_{up-out}(S, \tau; B_H) + \int_0^\tau e^{-r\omega} c_{down-out}(B_H, \tau - \omega; B_L) Q(\omega; S, B_H) d\omega, \quad (2.1a)$$

where

$$Q(\omega; S, B_H) = \frac{|\ln \frac{B_H}{S}|}{\sqrt{2\pi\sigma^2\omega^2}} \exp\left(-\frac{[\ln \frac{B_H}{S} - (r - q - \frac{\sigma^2}{2})\omega]^2}{2\sigma^2\omega}\right), \quad (2.1b)$$

$r$ ,  $q$  and  $\sigma$  are the riskless interest rate, dividend yield and volatility, respectively. The valuation of the expressions in Eq. (2.1a) gives the following analytical formula:

$$c_{seq}(S, \tau; B_H, B_L) = c_{van}(S, \tau) - \left(\frac{B_L}{B_H}\right)^{2\mu/\sigma^2} c_{van}\left(\left(\frac{B_L}{B_H}\right)^2 S, \tau\right), \quad (2.1c)$$

where  $\mu = r - q - \frac{\sigma^2}{2}$  is the risk neutralized drift rate for  $\ln S$  and  $c_{van}(S, \tau)$  is the price of the corresponding European vanilla call option.

It occurs that similar analytical price formula for  $c_{seq}(S; \tau; B_H, B_L)$  has been obtained by Li [10] using the reflection principle in restricted Brownian process. However, the idea of taking the reduced single barrier option as rebate upon breaching of the first barrier used in formulating Eq. (2.1a) is crucial in the construction of the numerical schemes for solving the discretely monitored sequential barrier options [see Eq. (2.7)].

## 2.2. Discretely monitored sequential barriers

Next, we would like to derive the analytical price formula for a European option with discretely monitored sequential barriers, where the upstream barrier  $B_H$  is the first barrier and the downstream barrier  $B_L$  is the second barrier. The sequential barrier option survives up to the expiration time if either (i) the first barrier is never breached, except possibly at the last monitoring instant, or (ii) the first barrier has been breached but the second barrier is never breached at all subsequent monitoring instants. Let  $t$  and  $T$  be the current time and expiration time, respectively, and  $t_1 < t_2 < \dots < t_n$  be the  $n$  monitoring instants between  $t$  and  $T$ . Let  $S_{t_i}(S_T)$  denote the asset price at time  $t_i(T)$ . The value of this discretely monitored sequential barrier call option is given by

$$\begin{aligned}
& c_{seq}(S, \tau; B_H, B_L, t_1, \dots, t_n) \\
= & e^{-r\tau} E(S_T - X) 1_{\{S_T > X, S_1, \dots, S_{n-1} < B_H\}} \\
& + e^{-r\tau} E(S_T - X) 1_{\{S_T > X, S_1, \dots, S_{n-2} < B_H, S_{n-1} > B_H, S_n > B_L\}} \\
& + \dots + e^{-r\tau} E(S_T - X) 1_{\{S_T > X, S_1 < B_H, S_2 > B_H, S_3, \dots, S_n > B_L\}} \\
& + e^{-r\tau} E(S_T - X) 1_{\{S_T > X, S_1 > B_H, S_2, \dots, S_n > B_L\}}, \tag{2.2}
\end{aligned}$$

where  $\tau = T - t$ ,  $1_{\{\cdot\}}$  is the indicator function and  $E$  denotes the expectation under the probability measure associated with the risk neutral asset price process with  $r - q$  as the drift rate. The first term corresponds to the case where the first barrier is not breached at  $t_1, \dots, t_{n-1}$ , the second term corresponds to the case where the first barrier is first breached at  $t_{n-1}$  but the second barrier is not breached at  $t_n, \dots$ , the last term corresponds to the case where the first barrier is breached at  $t_1$  but the second barrier is not breached at all subsequent monitoring instants.

The multi-dimensionality of the discretely monitored model arises from the calculations of the conditional expectation of the terminal payoff under the joint processes of the asset price ratios corresponding to overlapping time intervals. By direct evaluation of the above expectations (see [7] for the technique of expectation calculations used in their single barrier option model), the barrier option value is found to be

$$\begin{aligned}
& c_{seq}(S, \tau; B_H, B_L, t_1, \dots, t_n) \\
= & S e^{-q\tau} N_n(d_1^T, -d_1^1, -d_1^2, \dots, -d_1^{n-1}; \gamma) \\
& - e^{-r\tau} X N_n(d_2^T, -d_2^1, -d_2^2, \dots, -d_2^{n-1}; \gamma) \tag{2.3} \\
& + \sum_{j=1}^{n-1} \left\{ S e^{-q\tau} N_{n+1}(d_1^T, -d_1^1, \dots, -d_1^{j-1}, d_1^j, \tilde{d}_1^{j+1}, \dots, \tilde{d}_1^n; \Gamma^j) \right. \\
& \quad \left. - e^{-r\tau} X N_{n+1}(d_2^T, -d_2^1, \dots, -d_2^{j-1}, d_2^j, \tilde{d}_2^{j+1}, \dots, \tilde{d}_2^n; \Gamma^j) \right\},
\end{aligned}$$

where  $N_n$  is the  $n$ -dimensional cumulative normal distribution function,

$$d_1^T = \frac{\ln \frac{S}{X} + \left(r - q + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2^T = d_1^T - \sigma\sqrt{T - t}, \tag{2.4a}$$

$$\begin{aligned}
d_1^j &= \frac{\ln \frac{S}{B_H} + \left(r - q + \frac{\sigma^2}{2}\right)(t_j - t)}{\sigma\sqrt{t_j - t}}, \quad d_2^j = d_1^j - \sigma\sqrt{t_j - t}, \\
& j = 1, 2, \dots, n - 1, \tag{2.4b}
\end{aligned}$$

$$\begin{aligned}
\tilde{d}_1^j &= \frac{\ln \frac{S}{B_L} + \left(r - q + \frac{\sigma^2}{2}\right)(t_j - t)}{\sigma\sqrt{t_j - t}}, \quad \tilde{d}_2^j = \tilde{d}_1^j - \sigma\sqrt{t_j - t}, \\
& j = 2, 3, \dots, n. \tag{2.4c}
\end{aligned}$$

Further,  $\gamma$  is a  $n \times n$  correlation matrix and  $\Gamma^j, j = 1, 2, \dots, n-1$ , are  $(n+1) \times (n+1)$  correlation matrices. Since  $\gamma$  and  $\Gamma^j, j = 1, 2, \dots, n-1$  are symmetric matrices with unit diagonal entries, the matrices are well defined provided that their  $(k, \ell)^{\text{th}}$  entries,  $k < \ell$ , are specified. The  $(k, \ell)^{\text{th}}$  entry,  $k < \ell$ , in the correlation matrix  $\gamma$  is given by

$$\gamma_{k\ell} = \begin{cases} \sqrt{\frac{t_k - t}{t_\ell - t}} & \text{if } \ell \neq n \\ -\sqrt{\frac{t_k - t}{T - t}} & \text{if } \ell = n \end{cases}. \quad (2.5)$$

The form of the  $(k, \ell)^{\text{th}}$  entry,  $k < \ell$ , in the correlation matrix  $\Gamma^j$  depends on the ordering of  $k, \ell$  and  $j$  and whether  $\ell = n+1$  or not. The entries in the correlation matrix  $\Gamma^j$  are

$$\Gamma_{k\ell}^j = \begin{cases} \sqrt{\frac{t_k - t}{t_\ell - t}} & \text{if } j \leq k < \ell \text{ or } k < \ell < j, & \ell \neq n+1 \\ -\sqrt{\frac{t_k - t}{t_\ell - t}} & \text{if } k < j \leq \ell, & \ell \neq n+1 \\ -\sqrt{\frac{t_k - t}{T - t}} & \text{if } k < j, & \ell = n+1 \\ \sqrt{\frac{t_k - t}{T - t}} & \text{if } j \leq k, & \ell = n+1 \end{cases}. \quad (2.6)$$

The numerical valuation of  $N_n(\cdot)$  appearing in the analytic formula (2.3) (where  $n$  may take value beyond 100) can be very computationally demanding (see [12] for the quasi Monte Carlo method for the numerical valuation of related derivative models). Fortunately, the valuation problem can be reduced into succession of one-dimensional problems as shown in the following subsection.

### 2.3. Numerical schemes

The valuation algorithm for the pricing of discretely monitored sequential barrier call option can be constructed by a slight modification of the existing finite difference algorithm for discretely monitored barrier options.

Let  $\tau_1^*$  denote the last monitoring instant,  $\tau_2^*$  denote the second last monitoring instant, etc. Let  $k_j$  denote the number of time steps between  $\tau_{j-1}^*$  and  $\tau_j^*$ ,  $j = 1, 2, \dots$  (take  $\tau_0^* = 0$ ). It would be convenient to choose  $\tau_j^*, j = 1, 2, \dots$  to fall onto horizontal layers of nodes in the finite difference meshes by adjusting the time steps accordingly. If the time intervals between the monitoring instants are uniform, then the time step can be taken to be constant, and this also results the same  $k_j$  for different values of  $j$ . When  $k_j$ 's are chosen to be equal to one for all  $j$ , the calculations give the option values corresponding to continuous monitoring of the barriers. With regard to the placing of the barriers, it has been known that best accuracy can be achieved if the barrier is placed between two vertical columns of

nodes for the discretely monitored barrier options and exactly on a vertical column of nodes for the continuously monitored counterparts [2].

To initiate the calculations, we observe that the value of the present option equals to that of the corresponding vanilla counterpart at the last monitoring instant since no knock-out is possible between the last monitoring instant and the expiration time. One may use the Black-Scholes pricing formula for vanilla option to compute the option values at the nodes at the last monitoring instant  $\tau_1^*$ . This is recommended since it would avoid the possible deterioration of accuracy in subsequent calculations at later time steps arising from the discontinuity of the first order derivative of the terminal payoff function.

We perform the usual finite difference time marching procedure using the standard finite difference scheme since the present option behaves like a vanilla option between monitoring instants. When the time step corresponding to a monitoring instant is reached, the option values obtained from the time marching calculations at nodes below  $x = \ln B_H$  are kept unchanged while the option values at nodes above  $x = \ln B_H$  are set equal to the values of the corresponding down-and-out barrier option. This is because the first barrier  $B_H$  has been breached already so the present option with sequential barriers behaves like an ordinary down-and-out option. This follows the same approach of taking a down-and-out option as rebate upon breaching of the first barrier [see Eq. (2.1a)].

In summary, the finite difference scheme can be succinctly represented by

$$V_j^m = \begin{cases} U_j^m, & \text{if } x_j > \ln B_H \text{ and } m\Delta t = \tau_j^*, j \neq 1 \\ (p_u V_{j+1}^{m-1} + p_0 V_j^{m-1} + p_d V_{j-1}^{m-1})e^{-r\Delta t} & \text{otherwise} \end{cases} \quad (2.7)$$

where  $U_j^m$  is the option value of the down-and-out call at the same node. Actually,  $U_j^m$  and  $V_j^m$  represent the option values at node  $(j, m)$  with and without breaching of the first barrier, respectively. The coefficients  $p_u, p_0$  and  $p_d$  are given by

$$p_u = \frac{\mu + c}{2}, \quad p_d = \frac{\mu - c}{2} \quad \text{and} \quad p_0 = 1 - \mu, \quad (2.8)$$

where  $\mu = \sigma^2 \frac{\Delta t}{\Delta x^2}$  and  $c = \left(r - \frac{\sigma^2}{2}\right) \frac{\Delta t}{\Delta x}$ . Here,  $\Delta t$  and  $\Delta x$  are the time step and stepwidth used in the finite difference calculations, respectively.

First, a prior finite difference valuation procedure for the down-and-out barrier option values  $U_j^m$  is required. Once  $U_j^m$  are known, the time marching calculations for  $V_j^m$  resemble those for a plain vanilla option. Hence, the complexity of the numerical algorithm for the sequential barrier option is roughly equal to the sum of those for a down-and-out barrier

option and a plain vanilla option. The finite difference calculations for an one-asset option model require  $3NM$  multiplications and  $2NM$  additions, where  $N$  and  $M$  are the total number of time steps and the total number of spatial steps in the calculations, respectively. Typically,  $N \approx 500$  and  $M \approx 30$  are required in order to achieve percentage error in option values to be less than 0.1%.

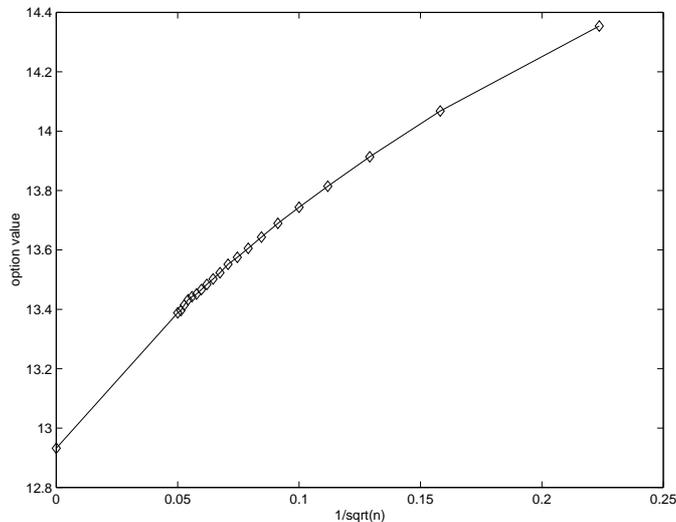
As a remark, similar approach of algorithm design can be applied to other types of barrier options, like the Parisian options, where the knock-out depends on the history of breaching of the barriers. In Parisian option calculations, one needs to add a counting index  $K$  as an extra dimension, where  $K$  counts the number of breaching of barriers occurred so far. In the present sequential barrier option calculations,  $K$  takes the value either 0 or 1, corresponding to no breaching or occurrence of breaching of the first barrier, respectively.

#### 2.4. Sample calculations

A numerical experiment was performed to verify the validity of the above proposed algorithm. Using scheme (2.7), the values of the European call options with discretely monitored sequential barriers were computed with varying number of monitoring instants  $n$ . The parameter values chosen for these call options are:

interest rate $r$	5%
volatility $\sigma$	25%
dividend yield $q$	0%
time to expiry $\tau$	1 (year)
spot asset price $S$	100 (\$)
strike price $X$	95 (\$)
first barrier level $B_H$	105 (\$)
second barrier level $B_L$	90 (\$)

For a fixed value of  $n$ , several numerical option values were obtained using different number of time steps  $k$  between successive monitoring instants. The Shanks transformation, which is a standard non-linear extrapolation technique, was applied to the numerical option values obtained with varying number of time steps so as to obtain the best estimate of the option value. These best estimated option values corresponding to different monitoring frequencies are plotted against  $1/\sqrt{n}$  in Figure 1. The continuously monitored case corresponds to  $n \rightarrow \infty$  or  $1/\sqrt{n} = 0$ , and the corresponding option value of 12.93 was obtained via numerical valuation of the analytic formula in Eq. (2.1c). The convergence trend of the option values with varying  $n$  to the limit corresponding to continuous monitoring is well revealed in Figure 1.



**FIG. 1** The plot reveals the convergence trend of the call option values with discretely monitored sequential barriers against  $1/\sqrt{n}$ . Here,  $n$  denotes the number of monitoring instants.

The exhibited rate of convergence of  $O\left(\frac{1}{\sqrt{n}}\right)$  of the discretely monitored barrier option values to the continuously monitored barrier option value (see Figure 1) does agree with similar results on continuity corrections for level-crossing probabilities of random walk. For example, Broadie *et al.* [3] showed in their continuity correction formula for discrete barrier options that one should shift the barrier away from  $S$  by a factor of  $\exp(\beta\sigma\sqrt{\Delta T})$ , where  $\Delta T$  is the uniform time interval between successive monitoring instants and  $\beta \approx 0.5826$ .

### 3. VALUATION ALGORITHMS FOR TWO-ASSET OPTION MODELS

In this section, we would like to illustrate the general approach of developing valuation algorithms for pricing multivariate path dependent options. We present the derivation method for the construction of explicit finite schemes of approximating the multi-dimensional diffusion type equations with cross-derivative terms. It is seen that the direct finite difference discretization of the cross-derivative term in the two-asset option price equation would lead to an explicit scheme with 9 points at the old time

level. On the other hand, the frequently used Hopscotch method [5] still involves 7 points at the old time level. The explicit finite difference scheme depicted below, which is derived using the Fourier method, uses a symmetric stencil which involves only 5 points at the old time level. At each lattice node, the 5-point scheme requires 5 multiplications and 4 additions, while the 7-point Hopscotch scheme requires 7 multiplications and 6 additions.

### 3.1. Derivation method based on the Fourier modes expansion

We consider the class of two-asset option pricing models where the dynamics of the underlying asset prices follow the lognormal distributions. Let  $V = V(S_1, S_2, \tau)$  denote the option price of a two-asset option, where  $S_1$  and  $S_2$  are the asset prices with  $\sigma_1$  and  $\sigma_2$  as their respective volatilities. Let  $\rho$  denote the correlation coefficient between the two lognormal processes. By writing

$$x_1 = \ln S_1, x_2 = \ln S_2 \quad \text{and} \quad v(x_1, x_2, \tau) = e^{-r\tau} V(S_1, S_2, \tau), \quad (3.1)$$

the governing equation for  $v = v(x_1, x_2, \tau)$  is given by

$$\begin{aligned} \frac{\partial v}{\partial \tau} = & \frac{\sigma_1^2}{2} \frac{\partial^2 v}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 v}{\partial x_2^2} \\ & + \left( r - q_1 - \frac{\sigma_1^2}{2} \right) \frac{\partial v}{\partial x_1} + \left( r - q_2 - \frac{\sigma_2^2}{2} \right) \frac{\partial v}{\partial x_2}, \end{aligned} \quad (3.2)$$

where  $q_1$  and  $q_2$  are the dividend yields of  $S_1$  and  $S_2$ , respectively. We would like to devise two-level explicit schemes of the form

$$\begin{aligned} v_{j,k}^{n+1} = & b_{1,1} v_{j+1,k+1}^n + b_{1,-1} v_{j+1,k-1}^n \\ & + b_{-1,1} v_{j-1,k+1}^n + b_{-1,-1} v_{j-1,k-1}^n + b_{0,0} v_{j,k}^n, \end{aligned} \quad (3.3)$$

where only 5 points at the old  $n^{\text{th}}$  time level are involved.

Suppose the eigenfunction solution of the continuous problem takes the form

$$v(x_1, x_2, \tau) = A(\tau) e^{ik_1 x_1} e^{ik_2 x_2}, \quad (3.4)$$

which is feasible for domains with periodic boundary conditions. By substituting the above solution into Eq. (3.2), we deduce that  $A(\tau)$  satisfies

$$\begin{aligned} \frac{dA(\tau)}{d\tau} = & \left[ -\frac{k_1^2 \sigma_1^2}{2} - \rho k_1 k_2 \sigma_1 \sigma_2 - \frac{k_2^2 \sigma_2^2}{2} \right. \\ & \left. + ik_1 \left( r - q_1 - \frac{\sigma_1^2}{2} \right) + ik_2 \left( r - q_2 - \frac{\sigma_2^2}{2} \right) \right] A(\tau). \end{aligned} \quad (3.5)$$

If we relate  $v(x_1, x_2, \tau + \Delta\tau)$  and  $v(x_1, x_2, \tau)$  by

$$v(x_1, x_2, \tau + \Delta\tau) = E(\Delta\tau; x_1, x_2)v(x_1, x_2, \tau), \quad (3.6)$$

where  $E(\Delta\tau; x_1, x_2)$  is the time evolution operator for the continuous problem, then it is seen that

$$E(\Delta\tau; x_1, x_2) = \exp\left(-\frac{\mu_1}{2}\xi_1^2 - \rho\sqrt{\mu_1\mu_2}\xi_1\xi_2 - \frac{\mu_2}{2}\xi_2^2 + ic_1\xi_1 + ic_2\xi_2\right), \quad (3.7)$$

where

$$\begin{aligned} \xi_1 &= k_1\Delta x_1, \quad \xi_2 = k_2\Delta x_2, \quad \mu_1 = \sigma_1^2 \frac{\Delta t}{\Delta x_1^2}, \quad \mu_2 = \sigma_2^2 \frac{\Delta t}{\Delta x_2^2}, \\ c_1 &= \left(r - q_1 - \frac{\sigma_1^2}{2}\right) \frac{\Delta t}{\Delta x_1} \quad \text{and} \quad c_2 = \left(r - q_2 - \frac{\sigma_2^2}{2}\right) \frac{\Delta t}{\Delta x_2}. \end{aligned} \quad (3.8)$$

The relative orders of magnitude of the stepwidth,  $\Delta x_1$  and  $\Delta x_2$ , and the time step  $\Delta t$  should observe  $O(\Delta t) = O(\Delta x_1^2) = O(\Delta x_2^2)$  in order to satisfy the stability requirements.

On the other hand, we assume that the finite difference scheme (3.3) admits periodic eigenfunction solution of the form  $A^n e^{ik_1 j \Delta x_1} e^{ik_2 k \Delta x_2}$ . Correspondingly, we define the discrete evolution operator  $K(\Delta\tau; \xi_1, \xi_2)$  by the relation

$$v_{j,k}^{n+1} = K(\Delta\tau; \xi_1, \xi_2)v_{j,k}^n. \quad (3.9)$$

By substituting the discrete eigenfunction solution into Eq. (3.3) and observing Eq. (3.9), we obtain

$$\begin{aligned} K(\Delta\tau; \xi_1, \xi_2) &= b_{0,0} + b_{1,1}e^{i(\xi_1+\xi_2)} + b_{1,-1}e^{i(\xi_1-\xi_2)} \\ &\quad + b_{-1,1}e^{i(-\xi_1+\xi_2)} + b_{-1,-1}e^{i(-\xi_1-\xi_2)}. \end{aligned} \quad (3.10)$$

The coefficients  $b_{0,0}, b_{1,1}, b_{1,-1}, b_{-1,1}$  and  $b_{-1,-1}$  are determined by expanding  $E(\Delta\tau; \xi_1, \xi_2)$  and  $K(\Delta\tau; \xi_1, \xi_2)$  in powers of  $\xi_1$  and  $\xi_2$  up to the second order terms and equating the corresponding like power terms of  $\xi_1$  and  $\xi_2$ . This leads to the following linear system of algebraic equations

$$\begin{aligned} b_{0,0} + b_{1,1} + b_{1,-1} + b_{-1,1} + b_{-1,-1} &= 1 \\ b_{1,1} + b_{1,-1} - b_{-1,1} - b_{-1,-1} &= c_1 \\ b_{1,1} - b_{1,-1} + b_{-1,1} - b_{-1,-1} &= c_2 \\ b_{1,1} + b_{1,-1} + b_{-1,1} + b_{-1,-1} &= \mu_1 + c_1^2 = \mu_2 + c_2^2 \\ b_{1,1} - b_{1,-1} - b_{-1,1} + b_{-1,-1} &= \rho\sqrt{\mu_1\mu_2} + c_1c_2. \end{aligned} \quad (3.11)$$

the solution of which gives the required coefficients in the explicit scheme (3.3). Note that there is a constraint on the relative magnitude of  $\Delta x_1$  and  $\Delta x_2$ , as dictated by the relation:  $\mu_1 + c_1^2 = \mu_2 + c_2^2$ .

The technique of assuming Fourier eigenfunction solutions and matching the corresponding like power terms in  $\xi_1$  and  $\xi_2$  in the expansion of the continuous and discrete evolution operators is known to have general applicability. All types of explicit and implicit schemes for solving two-asset or higher dimensional option models can be derived in this systematic manner.

### *Skew computational stencils*

When the numerical boundary conditions along the boundaries of the computational domain cannot be inferred from the partial differential equation formulation of the continuous problem, the artificial imposition of numerical boundary conditions may lead to substantial errors in the calculated option values. For example, in the finite difference calculations for callable bonds, Büttler [4] illustrated through extensive numerical experiments that the accuracy of the bond values depends sensibly on the choice of the imposed numerical boundary condition at the limiting zero value of the interest rate. In general, it is preferable to use a skew computational stencil along the boundary nodes rather than to adopt an inaccurately prescribed numerical boundary conditions.

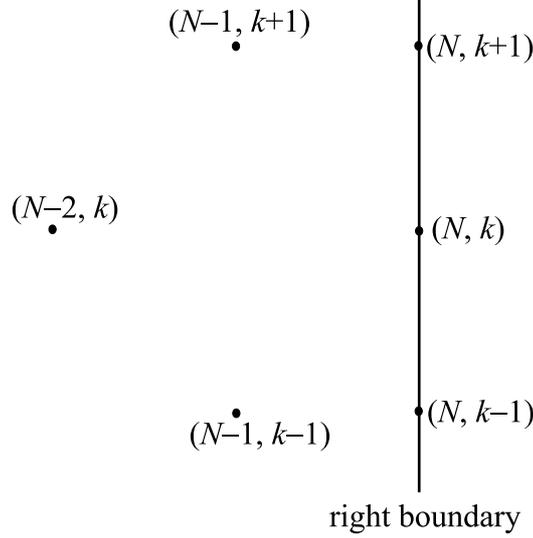
Consider the choice of the computational stencil along the far right boundary of a two-dimensional computational domain as shown in Figure 2. The corresponding two-level explicit scheme takes the form

$$\begin{aligned} v_{N,k}^{n+1} = & \alpha_{N-2,k} v_{N-2,k}^n + \alpha_{N-1,k+1} v_{N-1,k+1}^n + \alpha_{N-1,k-1} v_{N-1,k-1}^n \\ & + \alpha_{N,k+1} v_{N,k+1}^n + \alpha_{N,k} v_{N,k}^n + \alpha_{N,k-1} v_{N,k-1}^n. \end{aligned} \quad (3.12)$$

This skew stencil avoids the use of fictitious points beyond the right boundary of the computational domain, but at the expense of the loss of symmetry. The numerical scheme (3.12) approximates the governing equation (3.2), same as scheme (3.3) does, on the ground that the option value at node  $(N, k)$  still satisfies the governing equation. With non-symmetric stencil, 6 points instead of 5 points are required at the old time level since there are 6 relations to be satisfied. These relations are obtained by matching like power terms in  $\xi_1$  and  $\xi_2$  up to the second order and the coefficients must be summed to one. The corresponding linear system of algebraic equations for the coefficients in the numerical scheme is given by

$$\begin{aligned} \alpha_{N-2,k} + \alpha_{N-1,k+1} + \alpha_{N-1,k-1} + \alpha_{N,k+1} + \alpha_{N,k} + \alpha_{N,k-1} &= 1 \\ -2\alpha_{N-2,k} - \alpha_{N-1,k+1} - \alpha_{N-1,k-1} &= c_1 \\ \alpha_{N-1,k+1} + \alpha_{N,k+1} - \alpha_{N-1,k-1} - \alpha_{N,k-1} &= c_2 \end{aligned}$$

$$\begin{aligned}
\alpha_{N-1,k+1} + 4\alpha_{N-2,k} + \alpha_{N-1,k-1} &= \mu_1 + c_1^2 \\
\alpha_{N-1,k+1} + \alpha_{N,k+1} + \alpha_{N-1,k-1} + \alpha_{N,k-1} &= \mu_2 + c_2^2 \\
-\alpha_{N-1,k+1} - \alpha_{N-1,k-1} &= \rho\sqrt{\mu_1\mu_2} + c_1c_2.
\end{aligned} \tag{3.13}$$



**FIG. 2** Non-symmetric computational stencil along the far right boundary of a two-dimensional computational domain.

The same approach can be used to devise the modified numerical schemes for nodes along any boundary or at a corner of the computational domain. One drawback of adopting such skew discretized schemes along the boundaries is the possible loss of overall order of accuracy of the calculated option values with the loss of symmetry. This is because the leading truncation error terms are second order in  $\Delta x_1$  and  $\Delta x_2$  (or equivalently, first order in  $\Delta t$ ) when a symmetric stencil is used, but the error terms become first order in  $\Delta x_1$  and  $\Delta x_2$  when a skew stencil is adopted.

To illustrate the applicabilities of the above proposed numerical schemes, we performed sample calculations on two path dependent option models, namely, options with single external barrier and two-asset lookback options.

### 3.2. Options with single external barrier

The barrier option models with single external barrier are desirable for our purpose since closed form analytical price formulas are available when the barrier is monitored continuously (see [9]), thus making the comparison

of numerical accuracy of calculated option values feasible. Sample calculations for both continuously monitored and discretely monitored cases were performed using explicit scheme (3.12), together with various techniques of treatment of boundary conditions.

For the continuously monitored European call option with single external barrier, the governing equation for the option price is given by Eq. (3.2). The initial condition is

$$v(x_1, x_2, 0) = \max(e^{x_1} - X, 0), \quad -\infty < x_1 < \infty, x_2 > \ln B, \quad (3.14)$$

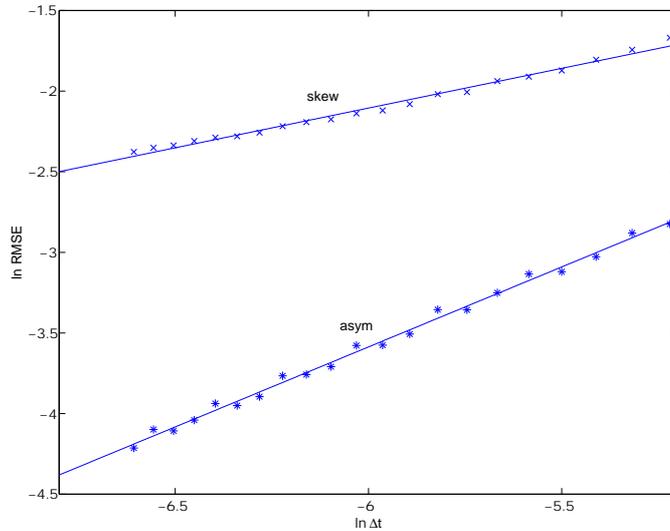
while the boundary conditions are

$$\begin{aligned} v(x_1, \ln B, \tau) &= 0, \quad \lim_{x_1 \rightarrow -\infty} v(x_1, x_2, \tau) = 0 \\ \lim_{x_2 \rightarrow \infty} v(x_1, x_2, \tau) &= e^{-r\tau} c_E(x_1, \tau), \quad \tau > 0, \end{aligned} \quad (3.15)$$

where  $X$  and  $B$  are the strike price and the barrier level, respectively, and  $c_E(x_1, \tau)$  is the price of the European vanilla counterpart. The boundary condition at the far field  $x_1 \rightarrow \infty$  can be quite tricky to be deduced. Kwok *et al.* managed to deduce the asymptotic boundary condition for the continuously monitored case (see Eqs. (32, 33), [9]). However, when the barrier is monitored discretely, the corresponding asymptotic formula for the far field boundary condition at  $x_1 \rightarrow \infty$  is not readily available.

For continuously monitored case, we compare the order of accuracy of the calculated option values obtained using (i) skew stencils along boundary nodes, and (ii) asymptotic formulas for boundary conditions. The parameter values chosen for the two-asset external barrier option models are:  $r = 5\%$ ,  $\sigma_1 = \sigma_2 = 20\%$ ,  $q_1 = q_2 = 0$ ,  $\tau = 0.5$ ,  $X = 20$ ,  $B = 15$ ,  $\rho = 0.5$ ,  $10 \leq S_1 \leq 80$ ,  $15 \leq S_2 \leq 95$ .

To access the order of accuracy of the calculated option values, we compute the root mean squared error (RMSE) by summing all squared errors at all nodes, taking the square root and dividing by the number of nodes. Figure 3 shows the plots of  $\ln$  RMSE against  $\ln \Delta t$  for option values obtained by both methods of treatment of the far field boundary conditions. The slopes of the plots reveal that the calculated option values using skew stencil and asymptotic formula are almost proportional to  $\sqrt{\Delta t}$  and  $\Delta t$ , respectively. When the time step  $\Delta t$  is decreased by a factor of one fourth, the errors of calculated option values are decreased by about half using skew stencil and decreased by about one fourth using asymptotic formula for boundary condition. The imposition of lower order discretized schemes along the boundaries of the computational domain reduces the overall order of accuracy of the numerical option values.

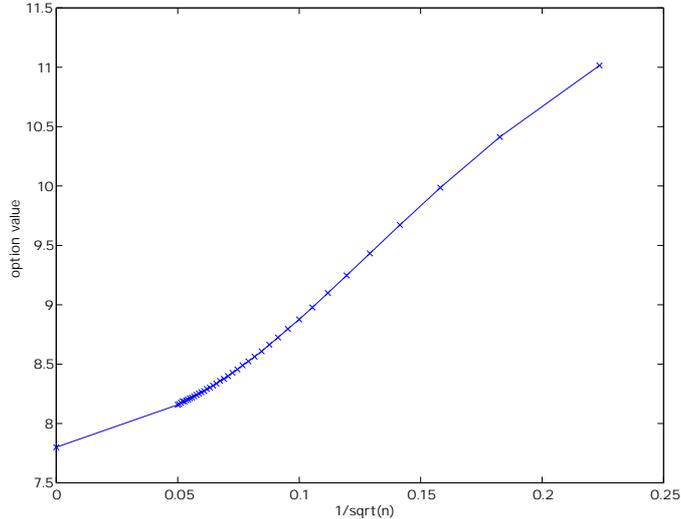


**FIG. 3** Plots of  $\ln$  RMSE against  $\ln \Delta t$  for option values obtained using the techniques of skew stencil and asymptotic formula. By regression calculations, the slopes of the upper line (skew stencil) and the lower line (asymptotic formula) are found to be 0.4930 and 0.9927, respectively.

For discretely monitored case, the skew discretized schemes must be used for boundary nodes since the asymptotic formulas for the boundary conditions are not available. In the numerical valuation of discretely monitored barrier options, the truncation of domain and the application of the barrier conditions are applied only at those time levels which correspond to monitoring instants. For nodes along the boundary of the computational domain whose values are not explicitly prescribed, we use the skew discretized schemes. We computed the values of discretely monitored external barrier options with varying number of monitoring instants  $n$  and plot the option values against  $1/\sqrt{n}$  in Figure 4. The parameter values chosen for the calculations are:  $r = 5\%$ ,  $\sigma_1 = \sigma_2 = 25\%$ ,  $q_1 = q_2 = 0$ ,  $\tau = 1$ ,  $X = 95$ ,  $B = 90$ ,  $\rho = 0.5$ ,  $S_1 = S_2 = 100$ . The values of the discretely monitored barrier options apparently converge to that of the continuously monitored counterpart as the number of monitoring instants tends to infinity.

The pricing algorithms for the external barrier option models can be extended to the pricing of contingent claim models of analyzing the credit risk of corporate debt issuers (see [11]). In these credit risk models, the stochastic state variables are the firm value and the interest rate; and the issuing

firm defaults when the firm value falls below some threshold value (down-and-out barrier). In the differential equation formulation of these models, one would encounter the difficulties of prescribing the boundary conditions at vanishing interest rate and exceedingly high firm value. The technique of adopting skew computational stencil along the domain boundaries could help avoid the prescription of artificial numerical boundary conditions.



**FIG. 4** The apparent trend of convergence of the option values of discretely monitored external barrier options to that of the continuously monitored counterpart is demonstrated.

### 3.3. Two-asset lookback option models

We consider the valuation of the European two-asset lookback option model where the terminal payoff involves the difference of one asset price and the extremum of another asset price. Let  $M_1$  denote the maximum of asset price  $S_1$  over the period  $[T_0, t]$ , where  $T_0$  is the starting time of the lookback period and  $t$  is the current time. The terminal payoff of this two-asset European lookback option at  $t = T$ , where  $T$  is the expiry date, is taken to be  $\max(M_1 - S_2, 0)$ . The governing equation for the lookback option price is known to be identical to that for the usual two-asset vanilla option models, except that the auxiliary conditions in the lookback option model involve an additional path dependent state variable,  $M_1$ . Hence, the present lookback option price  $V$  is a function of three state variables:  $S_1, S_2, M_1$  and time to expiry  $\tau$ . Note that  $S_1$  is defined only for  $S_1 \leq M_1$ . The boundary condition at  $S_1 = M_1$  is given by  $\left. \frac{\partial V}{\partial M_1} \right|_{S_1=M_1} = 0$ , using

the familiar argument that  $V$  should be insensitive to the change in  $M_1$  once  $S$  hits  $M_1$  (see [8]).

Let  $V(S_1, S_2, M_1, \tau)$  denote the value of the two-asset lookback option. Define the following similarity variables:

$$u = \frac{V}{S_1} e^{-q_1 \tau}, \quad z_1 = \ln \frac{M_1}{S_1} \quad \text{and} \quad z_2 = \ln \frac{S_2}{S_1}, \quad (3.16)$$

the governing equation for  $u = u(z_1, z_2, \tau)$  can be expressed as

$$\begin{aligned} \frac{\partial u}{\partial \tau} = & \frac{\sigma_1^2}{2} \frac{\partial^2 u}{\partial z_1^2} + \tilde{\rho} \sigma_1 \sigma_{12} \frac{\partial^2 u}{\partial z_1 \partial z_2} + \frac{\sigma_{12}^2}{2} \frac{\partial^2 u}{\partial z_2^2} \\ & + \left( q_1 - r - \frac{\sigma_1^2}{2} \right) \frac{\partial u}{\partial z_1} + \left( q_1 - q_2 - \frac{\sigma_{12}^2}{2} \right) \frac{\partial u}{\partial z_2}, \end{aligned} \quad (3.17)$$

where  $\rho$  is the correlation coefficient between  $S_1$  and  $S_2$ , and

$$\sigma_{12}^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \quad \text{and} \quad \tilde{\rho} = \frac{\sigma_1 - \rho\sigma_2}{\sigma_{12}}. \quad (3.18)$$

The auxiliary conditions are

$$u(z_1, z_2, 0) = \max(e^{z_1} - e^{z_2}, 0) \quad \text{and} \quad \frac{\partial u}{\partial z_1}(0, z_2, \tau) = 0. \quad (3.19)$$

Note that the dimension in the pricing model of the lookback option model can be reduced by one by using  $S_1$  as the numeraire. The choice of the similarity variables in Eq. (3.16) leads to the Neumann condition at  $z_1 = 0$ . Again, it is not so straightforward to prescribe the boundary conditions at the other boundaries of the problem domain.

To evaluate the value of this lookback option, He *et al.* [6] adopted the approach of finding the probability density of the maximum of one asset price process and another asset price process, then computing the discounted expectation of the terminal payoff by integrating the product of the density function and the terminal payoff. Suppose we define

$$\tilde{z}_1 = \ln \frac{M_1^T}{S_1}, \quad (3.20)$$

where  $M_1^T$  is the maximum of the price of asset one over  $[t, T]$ , then the lookback option value is given by

$$\begin{aligned} V = & e^{-r\tau} \int_0^\infty d\tilde{z}_1 \int_{-\infty}^\infty \max \left[ S_1 e^{\max(\tilde{z}_1, z_1)} - S_1 e^{z_2} \right] \\ & P[\tilde{z}_1(t) \in dz_1, z_2(t) \in dz_2] dz_2, \end{aligned} \quad (3.21)$$

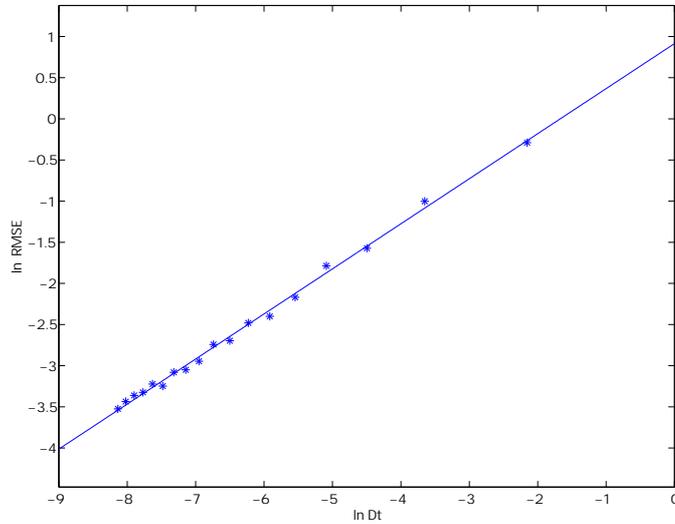
where  $z_1$  and  $z_2$  are defined in Eq. (3.16), and  $P[\tilde{z}_1(t) \in dz_1, z_2(t) \in dz_2]$  is the required joint density function. We manage to obtain the following price formula for the two-asset lookback option:

$$\begin{aligned} V(S_1, S_2, M_1, \tau) &= M_1 e^{-r\tau} N_2(k_1, \ell_1; \rho) - S_2 e^{-q_2\tau} N_2(k_2, \ell_2; \rho) \\ &\quad + S_1 e^{-q_1\tau} N_2(\tilde{k}_1, \tilde{\ell}_1; -\tilde{\rho}) - S_2 e^{-q_2\tau} N_2(\tilde{k}_2, \tilde{\ell}_2; -\tilde{\rho}) \\ &\quad + S_1 e^{-r\tau} \int_{z_1}^{\infty} e^{\frac{2(r-q_1)u}{\sigma_{12}}} N_2(k(u), \ell(u); \rho) du, \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} k_1 &= \frac{\ln \frac{M_1}{S_2} - \left(r - q_2 - \frac{\sigma_2^2}{2}\right) \tau}{\sigma_2 \sqrt{\tau}}, & k_2 &= k_1 - \sigma_2 \sqrt{\tau}, \\ \ell_1 &= \frac{\ln \frac{M_1}{S_1} - \left(r - q_1 - \frac{\sigma_1^2}{2}\right) \tau}{\sigma_1 \sqrt{\tau}}, & \ell_2 &= \ell_1 - \rho \sigma_2 \sqrt{\tau}, \\ \tilde{k}_1 &= \frac{\ln \frac{S_1}{S_2} - \left(q_2 - q_1 - \frac{\sigma_{12}^2}{2}\right) \tau}{\sigma_{12} \sqrt{\tau}}, & \tilde{k}_2 &= \tilde{k}_1 - \sigma_{12} \sqrt{\tau}, \\ \tilde{\ell}_1 &= \frac{-\ln \frac{M_1}{S_1} + \left(r - q_1 + \frac{\sigma_1^2}{2}\right) \tau}{\sigma_1 \sqrt{\tau}}, & \tilde{\ell}_2 &= \tilde{\ell}_1 + \tilde{\rho} \sigma_{12} \sqrt{\tau}, \\ k(u) &= \frac{-u - \left(r - q_1 - \frac{\sigma_1^2}{2}\right) \tau}{\sigma_1 \sqrt{\tau}}, \\ \ell(u) &= \frac{\ln \frac{S_1}{S_2} + u - \left(r - q_2 - \frac{\sigma_2^2}{2}\right) \tau}{\sigma_2 \sqrt{\tau}} - 2\rho \frac{u}{\sigma_1 \sqrt{\tau}}. \end{aligned} \quad (3.23)$$

We performed the computation of the lookback option values using the finite difference scheme, together with the adoption of skew discretized schemes along the computational boundaries other than  $z_1 = 0$ . The parameter values chosen for the two-asset lookback option are:  $r = 5\%$ ,  $\sigma_1 = \sigma_2 = 20\%$ ,  $\tau = 1$ ,  $\rho = 0.5$ ,  $q_1 = q_2 = 0$ ,  $z_1 = z_2 = 0.5$ . By valuation of the analytical formula in Eq. (3.22), the lookback option values is found to be 0.0992. In order to achieve percentage error of about 0.1%, our finite difference calculations used 450 time steps and 30 grid points in each spatial dimension. The plot of  $\ln$  RMSE against  $\ln \Delta t$  in Figure 5 reveals the square root rate of convergence in  $\Delta t$  of the numerical lookback option values. This is not surprising since the discretization along the boundaries of the computational domain other than  $z_1 = 0$  is accurate only to  $O(\Delta z_1)$  and  $O(\Delta z_2)$ . As we observe  $O(\Delta t) = O(\Delta z_1^2) = O(\Delta z_2^2)$  in lieu of stability requirements, so square root rate of convergence in  $\Delta t$  results.



**FIG. 5** Plot of  $\ln$  RMSE of the numerical option values of the two-asset European lookback option against  $\ln \Delta t$ . The slope of the regression line is found to be 0.5476, thus confirming roughly square root rate of convergence in  $\Delta t$ .

The finite difference algorithms for pricing other lookback options with payoff structures like  $\max(S_{\max} - S_{\min} - X, 0)$ ,  $\max(S_{1,\max} - S_{2,\min} - X, 0)$ , etc. can be developed by following the above approach (see [6] for an alternative approach, where He *et al.* computed the discounted expectation of the terminal payoff by the direct numerical integration of the expectation integrals). It is always advisable to normalize the extremum quantities by the asset prices, that is, the asset prices are used as numeraire.

#### 4. CONCLUSIONS

Several valuation techniques for the pricing of multivariate path dependent options have been illustrated through three prototype option models, namely, options with sequential barriers, options with single external barrier and two-asset lookback options. The specific nature of the path dependent feature of the option model, for example, the discrete monitoring of the barrier, extremum of asset prices over a time period, may increase the dimensionality of an option model beyond the number of underlying assets in the option.

For practitioners in the financial markets, they prefer option pricing algorithms that possess the characteristics of general applicabilities and ease of design. The various types of valuation algorithms presented in this paper are meant to achieve the above objectives.

For options with discretely monitored sequential barriers, though the analytical price formula possesses analytical elegance, it is almost rendered useless for valuation since it involves the high dimensional cumulative distribution functions. Fortunately, the corresponding numerical algorithm can reduce the valuation problem into succession of one-dimensional problems. It turns out that the complexity of calculations is roughly equal to the sum of those for a plain vanilla option and a single barrier option.

As illustrated by the sample calculations on options with single external barrier, it is seen that in situations where the boundary conditions are not explicitly specified in the option models, the choice of skew computational stencils at nodes along the computational boundaries provides an easy route to avoid the artificial imposition of numerical boundary conditions. However, the loss of symmetry in the skew stencils may lead to the loss of order of accuracy in the calculated option values.

For the lookback options, the use of the asset prices as numeraires leads to the reduction of dimensionality of the model. The finite difference approach again exhibits its competitiveness in its ease of design and programming efforts.

The search for better designed pricing algorithms for more complicated multivariate path dependent options remains to be a challenging task for finance researchers.

## REFERENCES

1. F. Black and M. Scholes, "The pricing of option and corporate liabilities," *Journal of Political Economy*, **81** (1973) 637-659.
2. P. Boyle and Y. Tian, "An explicit finite difference approach to the pricing of barrier options," *Applied Mathematical Finance*, **5** (1998) 6-14.
3. M. Broadie, P. Glasserman and S. Kou, "A continuity corrections for discrete barrier options," *Mathematical Finance*, **7** (1997) 325-349.
4. H. J. Büttler, "Evaluation of callable bonds: finite difference methods, stability and accuracy," *Economic Journal*, **105** (1995) 374-384.
5. A. R. Gourlay and S. McKee, "The construction of hopscotch methods for parabolic and elliptic equations in two space dimensions with a mixed derivative," *Journal of Computational and Applied Mathematics*, **3** (1977) 201-206.
6. H. He, W.P. Keirstead and J. Rebholz, "Double lookbacks," *Mathematical Finance*, **8** (1998) 201-228.
7. R.C. Heynen and H.M. Kat, "Discrete partial barrier options with a moving barrier," *Journal of Financial Engineering*, **5** (1996) 199-209.
8. Y.K. Kwok, *Mathematical models of financial derivatives*, Springer, Singapore (1998)

9. Y.K. Kwok, L. Wu and H. Yu, "Pricing multi-asset options with an external barrier," *International Journal of Theoretical and Applied Finance*, **1** (1998) 523-541.
10. A. Li, "The pricing of double barrier options and their variations," *Advances in Futures and Options Research*, **10** (1999) 17-41.
11. F. Longstaff and E. Schwartz, "A simple approach to valuing risky fixed and floating rate debt," *Journal of Finance*, **50** (1995) 789-819.
12. S. Paskov and J.F. Traub, "Faster valuation of financial derivatives," *Journal of Portfolio Management*, (1995) 113-120.
13. J. Sidenius, "Double barrier options: valuation by path counting," *Journal of Computational Finance*, **1** (1998) 63-79.