Quanto lookback options

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Abstract. The lookback feature in a quanto option refers to the payoff structure where the terminal payoff of the quanto option depends on the realized extreme value of either the stock price or the exchange rate. In this paper, we study the pricing models of European and American lookback option with the quanto feature. The analytic price formulas for two types of European style quanto lookback options are derived. The success of the analytic tractability of these quanto lookback options depends on the availability of a succinct analytic representation of the joint density function of the extreme value and terminal value of the stock price and exchange rate. We also analyze the early exercise policies and pricing behaviors of the quanto lookback option with the American feature. The early exercise boundaries of these American quanto lookback options exhibit properties that are distinctive from other two-state American option models.

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1. Introduction

Lookback options are contingent claims whose payoff depends on the extreme value of the underlying asset price process realized over a specified period of time within the life of the option. The term "quanto" is an abbreviation for "quantity adjusted", and it refers to the feature where the payoff of an option is determined by the financial price or index in one currency but the actual payout is realized in another currency.

We examine the pricing models of European and American quanto lookback options whose payoff depends on the joint process of the stock price and exchange rate. In our option valuation framework, we assume lognormal process for the underlying stock price and exchange rate, and continuous monitoring of these two stochastic state variables. For the *joint quanto lookback options*, the lookback feature is applied on the stock price process and the exchange rate in the payoff is chosen to be the maximum of a pre-determined floor value and the terminal value at expiry. When the lookback feature is applied on the exchange rate process, this leads to the *maximum rate quanto lookback option*. Here, the exchange rate in the payoff is given by the realized maximum value over some monitoring period.

The pricing of lookback options poses interesting mathematical challenges. The analytic price formulas for European one-asset lookback options have been systematically derived by Goldman *et al.* (1979), and Conze and Viswanathan (1991). For two-state European lookback options, He *et al.* (1998) and Babsiri and Noel (1998) have obtained analytic expressions of the joint probability density functions of the extreme and terminal values of the prices of the underlying assets. However, due to the analytic complexity in their analytic expressions for the density functions, they did not proceed further in evaluating the discounted expectation integrals. Instead, they computed the lookback option prices via numerical integration of the discounted expectation integrals or Monte Carlo simulation.

In this paper, we derive the analytic price formulas for two types of European quanto lookback options under the lognormal assumption of the asset price process. The success of the analytic tractability of these quanto lookback options lies on our derivation of a new succinct representation of the joint density function of the extreme value and terminal value of the stock price and the exchange rate. In the derivation procedure, the standard quanto pre-washing techniques for dealing with quanto option models are used. With the availability of the closed form price formulas, we are able to comprehend various contributing factors to the value of these quanto lookback options.

In the next section, we summarize the quanto pre-washing techniques for dealing with the quanto feature in the pricing models, and present the probability density functions that involve the joint processes for the maximum value and the terminal value of the stock price and exchange rate. We then derive the analytic price formulas of the European style joint quanto lookback option and maximum rate quanto lookback option. In Section 3, we analyze the early exercise policies and pricing behaviors of these two types of quanto lookback options with the American feature. The properties of the optimal exercise boundaries are verified through numerical experiments. The paper is ended with conclusive remarks in the last section.

2. European quanto lookback options

In this section, we derive the analytic price formulas of two types of European quanto lookback options, where the lookback feature is applied on the exchange rate or the stock price. The usual assumptions of the Black-Scholes option pricing framework are adopted in this paper. Let F_t denote the exchange rate at time t, which is defined as the domestic currency price of one unit of foreign currency. Let r_d and r_f denote the constant domestic and foreign riskless interest rates, respectively. In the risk neutralized domestic currency world, the stochastic process of F_t is assumed to be governed by

$$\frac{dF_t}{F_t} = (r_d - r_f)dt + \sigma_F dZ_F, \tag{2.1}$$

where σ_F is the volatility of F and dZ_F is the Wiener process. In the foreign currency world, the stochastic process for the risk neutralized stock price process S_t is assumed to follow

$$\frac{dS_t}{S_t} = (r_f - q)dt + \sigma_S dZ_S, \tag{2.2}$$

where σ_S and q are the volatility and dividend yield of S, respectively, and dZ_S is the Wiener process. By applying the standard quanto prewashing technique [see Dravid *et al.* for a thorough discussion of the technique], the risk neutralized drift rate of S_t in the domestic currency world is given by

$$\delta_S^d = r_f - q - \rho\sigma_S\sigma_F, \tag{2.3}$$

where ρ is the correlation coefficient between dZ_S and dZ_F, with $\rho dt = dZ_S dZ_F$.

We consider the pricing models of two types of European quanto lookback options whose terminal payoff functions in the domestic currency world are given by

(i) Quanto call option with maximum exchange rate

$$V_{max}(S, F, T) = F_{max}^{[T_0,T]}(S_T - K)^+, \tag{2.4a}$$

where $F_{max}^{[T_0,T]}$ is the realized maxima of the exchange rate F over the time period $[T_0, T]$, and K is the strike price in foreign currency. Here, $x^+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$.

(ii) Joint quanto fixed strike lookback call option

$$V_{joint}(S, F, T; F_c) = \max(F_c, F_T)(S_{max}^{[T_0,T]} - K)^+, \tag{2.4b}$$

where F_c is some pre-specified constant exchange rate.

2.1 Quanto call option with maximum exchange rate

We assume that the current time lies within the period $[T_0, T]$ for monitoring the maximum value of the exchange rate. For convenience, we take the current time to be the zeroth time so that $T_0 < 0 < T$. We define the following unit variance stochastic normal variables

$$X_t = \frac{1}{\sigma_S}\ln\frac{S_t}{S} \quad \text{and} \quad Y_t = \frac{1}{\sigma_F}\ln\frac{F_t}{F}, \quad t > 0, \tag{2.5}$$

where S and F are the current stock price and exchange rate, respectively. In the domestic currency world, the risk neutralized drift rates of X_t and Y_t are given by

$$\mu_X = \frac{r_f - q - \rho\sigma_S\sigma_F - \frac{\sigma_S^2}{2}}{\sigma_S} \quad \text{and} \quad \mu_Y = \frac{r_d - r_f - \frac{\sigma_F^2}{2}}{\sigma_F}, \tag{2.6}$$

respectively. In addition, we define the stochastic random variable M_t to be the logarithm of the normalized maximum value of the exchange rate over the period $[0, t]$

$$M_t = \frac{1}{\sigma_F} \ln \frac{F_{max}^{[0,t]}}{F}. \tag{2.7}$$

Also, we denote the corresponding quantity for the realized maximum value over the earlier period $[T_0, 0]$ by $M_0 = \dfrac{1}{\sigma_F} \ln \dfrac{F_{max}^{[T_0,0]}}{F}$. In terms of M_t and X_t defined above, the terminal payoff of the quanto call option with maximum exchange rate can be expressed as

$$F_{max}^{[T_0,T]}(S_T - K)^+ = Fe^{\sigma_F \max(M_0, M_T)}(Se^{\sigma_S X_T} - K)^+. \tag{2.8}$$

The value of this European maximum rate quanto call at the current time is given by

$$V_{max} = e^{-r_d T} \int_{-\infty}^{\infty} \int_0^{\infty} Fe^{\sigma_F max(M_0, m)}(Se^{\sigma_S x} - K)^+ f_{max}(x, m, T)\, dmdx, \tag{2.9}$$

where $f_{max}(x, m, T)$ is the joint density function of X_T and M_T.

We present the analytic representation of $f_{max}(x, m, T)$ in Theorem 1. Subsequently, the analytic price formula for V_{max} is given in Theorem 2. As a remark, He *et al.* (1998) obtained an alternative analytic representation on $f_{max}(x, m, T)$ through a tedious procedure (see Theorem 2.3 in their paper). Using their analytic result, it is almost insurmountable to obtain an analytic price formula for any two-state semi-lookback option.

Theorem 1

The density function $f_{max}(x, m, t)$ is given by

$$f_{max}(x, m, t) = \frac{\partial G_{max}}{\partial m}(x, m, t) \tag{2.10}$$

where

$$G_{max}(x, m, t) = \int_{-\infty}^{m} g_{max}(x, y, t; m)\, dy \tag{2.11a}$$

$$g_{max}(x, y, t; m) = \phi_2(\widetilde{x}, \widetilde{y}, t; \rho) - e^{2\mu_Y m}\phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, t; \rho), \quad m > 0. \tag{2.11b}$$

Here, $\widetilde{x} = x - \mu_X t$ and $\widetilde{y} = y - \mu_Y t$, where μ_X and μ_Y are defined in Eq. (2.6), ρ is the correlation coefficient between dZ_S and dZ_F, and

$$\phi_2(\widetilde{x}, \widetilde{y}, t; \rho) = \frac{1}{2\pi t\sqrt{1 - \rho^2}} \exp\left(-\frac{\widetilde{x}^2 - 2\rho\widetilde{x}\widetilde{y} + \widetilde{y}^2}{2(1 - \rho^2)t}\right). \tag{2.11c}$$

Hence, $\phi_2(\widetilde{x}, \widetilde{y}, t; \rho)$ is the bivariate normal density function with zero means and unit variance rates.

Theorem 2

The analytic price formula for the European maximum exchange rate quanto lookback call option is given by

$$\begin{aligned} V_{max} = {} & F_{max}^{[T_0,0]} \left[e^{-(r_d-\delta_S^d)T} S N_2(d_1, -e_1; -\rho) - e^{-r_d T} K N_2(d_2, -e_2; -\rho) \right] \\ & + F \left[e^{-(r_f-\delta_S^d)T} S N_2(\widehat{d}_1, \widehat{e}_1; \rho) - e^{-r_f T} K N_2(\widehat{d}_2, \widehat{e}_2; \rho) \right] \\ & + F\sigma_F \int_{M_0}^{\infty} e^{(\sigma_F + 2\mu_Y)m} \left[e^{-(r_d-\delta_S^d)T} S N_2(\widetilde{d}_1, -\widetilde{e}_1; -\rho) - e^{-r_d T} K N_2(\widetilde{d}_2, -\widetilde{e}_2; -\rho) \right] dm, \end{aligned} \tag{2.12}$$

where $N_2(x, y; \rho)$ is the standard bivariate distribution function with zero means and unit variances, and

$$\begin{aligned} d_2 &= \frac{\ln \frac{S}{K} + \mu_X \sigma_S T}{\sigma_S \sqrt{T}}, & e_2 &= \frac{\ln \frac{F}{F_{max}^{[T_0,0]}} + \mu_Y \sigma_F T}{\sigma_F \sqrt{T}}, \\ d_1 &= d_2 + \sigma_S \sqrt{T}, & e_1 &= e_2 + \rho\sigma_S \sqrt{T}, \\ \widehat{d}_2 &= d_2 + \rho\sigma_F \sqrt{T}, & \widehat{e}_2 &= e_2 + \sigma_F \sqrt{T}, \\ \widehat{d}_1 &= \widehat{d}_2 + \sigma_S \sqrt{T}, & \widehat{e}_1 &= \widehat{e}_2 + \rho\sigma_S \sqrt{T}, \\ \widetilde{d}_2 &= d_2 + \frac{2\rho m}{\sqrt{T}}, & \widetilde{e}_2 &= \frac{m + \mu_Y T}{\sqrt{T}}, \\ \widetilde{d}_1 &= \widetilde{d}_2 + \sigma_S \sqrt{T}, & \widetilde{e}_1 &= \widetilde{e}_2 + \rho\sigma_S \sqrt{T}. \end{aligned} \tag{2.13}$$

The price formula V_{max} consists of three terms. The first term gives the contribution to the option value that is conditional on $F_{max}^{[T_0,T]} = F_{max}^{[T_0,0]}$ (this corresponds to no updated maximum value on F to be realized over the future period $[0,T]$) and $S_T \geq K$, while the second term corresponds to $F_{max}^{[T_0,0]} < F_{max}^{[T_0,T]}$ and $S_T \geq K$. The last term gives the value of the bonus of potential upward adjustment on the realized value of the exchange rate whenever a new maximum value is reached.

Zero derivative condition at $F = F_{max}^{[T_0,0]}$

When the current value of exchange rate F happens to be at the realized maximum value $F_{max}^{[T_0,0]}$, should the option price be insensitive to infinitesimal changes in $F_{max}^{[T_0,0]}$? Mathematically, this is equivalent to ask whether $\left.\frac{\partial V_{max}}{\partial M_0}\right|_{M_0=0} = 0$. This result can be deduced easily by computing $\frac{\partial V_{max}}{\partial M_0}$ directly using the integral representation of V_{max} in Eq. (2.9) (see the Appendix).

The zero derivative condition at $F_{max}^{[T_0,0]}$ is important in the design of the finite difference algorithm for the numerical solution of the quanto lookback option. This is because the full prescription of the boundary conditions of the option model is required in the construction of the finite difference scheme.

2.2 Joint quanto fixed strike lookback call option

For the joint quanto fixed strike lookback call option, the maximum value is monitored continuously on the stock price process S_t. Accordingly, we define the stochastic random variable U_t to be the logarithm of the normalized maximum value over the period $[0, t]$ of the stock price, that is,

$$U_t = \frac{1}{\sigma_S} \ln \frac{S_{max}^{[0,t]}}{S}, \tag{2.14}$$

and denote the corresponding quantity for the realized maximum value over the earlier period $[T_0, 0]$ by $U_0 = \dfrac{1}{\sigma_S} \ln \dfrac{S_{max}^{[T_0,0]}}{S}$. In terms of U_T, U_0 and Y_T, the terminal payoff of the joint quanto lookback call can be expressed as

$$\begin{aligned} &\max(F_c, F_T)(S_{max}^{[T_0,T]} - K)^+ \\ &= \begin{cases} F_c(Se^{\sigma_S U_0} - K)^+ & \text{if } F_c \geq F_T \text{ and } S_{max}^{[T_0,0]} \geq S_{max}^{[0,T]} \\ F_c(Se^{\sigma_S U_T} - K)^+ & \text{if } F_c \geq F_T \text{ and } S_{max}^{[0,T]} > S_{max}^{[T_0,0]} \\ Fe^{\sigma_F Y_T}(Se^{\sigma_S U_0} - K)^+ & \text{if } F_T > F_c \text{ and } S_{max}^{[T_0,0]} \geq S_{max}^{[0,T]} \\ Fe^{\sigma_F Y_T}(Se^{\sigma_S U_T} - K)^+ & \text{if } F_T > F_c \text{ and } S_{max}^{[0,T]} > S_{max}^{[T_0,0]} \end{cases}. \end{aligned} \tag{2.15}$$

By following similar derivation procedure as that in Theorem 1, the density function of the joint process of Y_T and U_T is given by $\dfrac{\partial G_{joint}}{\partial u}(y, u, T)$, where

$$G_{joint}(y, u, T) = \int_{-\infty}^{u} g_{joint}(x, y, T; u)\, dx \tag{2.16}$$

and

$$g_{joint}(x, y, T; u) = \phi_2(\widetilde{x}, \widetilde{y}, T; \rho) - e^{2\mu_X u} \phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho), \quad u > 0. \tag{2.17}$$

The form of the analytic price formula of the joint quanto lookback option depends on the sign of $S_{max}^{[T_0,0]} - K$. When the option is currently in-the-money or at-the-money (corresponding to $S_{max}^{[T_0,0]} - K \geq 0$), it is guaranteed to expire in-the-money. On the other hand, when $S_{max}^{[T_0,0]} - K < 0$, the option will expire out-of-the-money when $S_{max}^{[T_0,0]} > S_{max}^{[0,T]}$. We derive the price formula of the joint quanto lookback call under the following two cases:

1. $S_{max}^{[T_0,0]} < K$ (currently out-of-the-money)

$$\begin{aligned} V_{joint} = e^{-r_d T} \Bigg\{ & F_c \int_{-\infty}^{\frac{1}{\sigma_F} \ln \frac{F_c}{F}} \int_{\frac{1}{\sigma_S} \ln \frac{K}{S}}^{\infty} (Se^{\sigma_S u} - K) \frac{\partial G_{joint}}{\partial u}(y, u, T)\, du dy \\ & + F \int_{\frac{1}{\sigma_F} \ln \frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S} \ln \frac{K}{S}}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K) \frac{\partial G_{joint}}{\partial u}(y, u, T)\, du dy \Bigg\}. \end{aligned} \tag{2.18}$$

2. $S_{max}^{[T_0,0]} \geq K$ (currently in-the-money or at-the-money)

$$
\begin{aligned}
V_{joint} = e^{-r_d T} \Bigg\{ & F_c(S_{max}^{[T_0,0]} - K) \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_0^{U_0} \frac{\partial G_{joint}}{\partial u}(y,u,T)\, dudy \\
+\ & F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{U_0}^{\infty} (Se^{\sigma_S u} - K) \frac{\partial G_{joint}}{\partial u}(y,u,T)\, dudy \\
+\ & F(S_{max}^{[T_0,0]} - K) \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F_0}}^{\infty} \int_0^{U_0} e^{\sigma_F y} \frac{\partial G_{joint}}{\partial u}(y,u,T)\, dudy \\
+\ & F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F_0}}^{\infty} \int_{U_0}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K) \frac{\partial G_{joint}}{\partial u}(y,u,T)\, dudy \Bigg\}. \qquad (2.19)
\end{aligned}
$$

Theorem 3

The analytic price formula of the joint quanto fixed strike lookback call option is given by

1. $S_{max}^{[T_0,0]} < K$

$$
\begin{aligned}
V_{joint} = {} & e^{-r_d T} F_c \left[Se^{\delta_S^d T} N_2(d_1, -f_1; -\rho) - K N_2(d_2, -f_2; -\rho) \right] \\
+\ & e^{-r_f T} F \left[Se^{\delta_S^d T} N_2(\widehat{d}_1, \widehat{f}_1; \rho) - K N_2(\widehat{d}_2, \widehat{f}_2; \rho) \right] \\
+\ & e^{-r_d T} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\
& \left[F_c S N_2(-\overline{e}_2, -\widetilde{f}_2; \rho) + F S N_2(-\overline{e}_1, \widetilde{f}_1; -\rho) \right] du, \qquad (2.20)
\end{aligned}
$$

where $d_1, d_2, \widehat{d}_1$ and $\widehat{d}_2$ are defined in Eq. (2.13), and

$$
\begin{aligned}
f_2 &= \frac{\ln\frac{F}{F_c} + \mu_Y \sigma_F T}{\sigma_F \sqrt{T}}, & f_1 &= f_2 + \rho\sigma_S\sqrt{T}, \\
\widehat{f}_2 &= f_2 + \sigma_F\sqrt{T}, & \widehat{f}_1 &= \widehat{f}_2 + \rho\sigma_S\sqrt{T}, \\
\widetilde{f}_2 &= f_2 + \frac{2\rho u}{\sqrt{T}}, & \widetilde{f}_1 &= \widetilde{f}_2 + \sigma_F\sqrt{T}, \\
\overline{e}_2 &= \frac{u + \mu_X T}{\sqrt{T}}, & \overline{e}_1 &= \overline{e}_2 + \rho\sigma_F\sqrt{T}. \qquad (2.21)
\end{aligned}
$$

The first two terms in Eq. (2.20) resemble closely to the price formula for the joint quanto European call option (Kwok and Wong, 2000), while the last term can be interpreted as the premium for potential upward adjustment on the realized maximum value of the stock price.

2. $S_{max}^{[T_0,0]} \geq K$

$$\begin{aligned} V_{joint} = & \; e^{-r_dT}(S_{max}^{[T_0,0]} - K)\left[F_c N_2(-d_2^M, -f_2; \rho) + F N_2(-\widehat{d}_2^M, \widehat{f}_2; -\rho)\right] \\ & + \; e^{-r_dT} F_c \left[S e^{\delta_S^d T} N_2(d_1^M, -f_1; -\rho) - K N_2(d_2^M, -f_2; -\rho)\right] \\ & + \; e^{-r_fT} F \left[S e^{\delta_S^d T} N_2(\widehat{d}_1^M, \widehat{f}_1; \rho) - K N_2(\widehat{d}_2^M, \widehat{f}_2; \rho)\right] \\ & + \; e^{-r_dT} \int_{U_0}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\ & \qquad \left[F_c S N_2(-\overline{e}_2, -\widetilde{f}_2; \rho) + F S N_2(-\overline{e}_1, \widetilde{f}_1; -\rho)\right] \, du, \end{aligned} \tag{2.22}$$

where $f_1, f_2, \widehat{f}_1, \widehat{f}_2, \widetilde{f}_1, \widetilde{f}_2, \overline{e}_1$ and $\overline{e}_2$ are defined in Eq. (2.21), and

$$\begin{aligned} d_2^M &= \frac{\ln \frac{S}{S_{max}^{[T_0,0]}} + \mu_X \sigma_S T}{\sigma_S \sqrt{T}}, & d_1^M &= d_2^M + \sigma_S \sqrt{T}, \\ \widehat{d}_2^M &= d_2^M + \rho \sigma_F \sqrt{T}, & \widehat{d}_1^M &= \widehat{d}_2^M + \sigma_S \sqrt{T}. \end{aligned} \tag{2.23}$$

The first term corresponds to the case where $S_{max}^{[T_0,0]} > K$ and conditional on no updated maximum value on S to be realized over the future period $[0, T]$. The second, third and fourth terms are similar to those in Eq. (2.20) except that the strike price K is replaced by $S_{max}^{[T_0,0]}$.

3. American quanto lookback options

The characteristics of the early exercise regions and optimal early exercise policies of American options on several risky assets are known to depend sensibly on the payoff structures of the options. Broadie and Detemple (1996) and Villeneuve (1999) provided some interesting results on the characterization of the early exercise regions of American extremum options and spread options. Except for the perpetual American options with very simple payoff structures, like perpetual Margrabe option and perpetual zero-strike maximum option [see Gerber and Shiu (1996)], it is not feasible to obtain analytic price formulas for multi-state American options. At best, we may obtain the analytic representation of the early exercise premium in terms of an integral that involves the exercise boundary function. The early exercise boundary is then solved via the solution of an integral equation.

It would be interesting to examine how the lookback feature interacts with the American early exercise feature. One example of an American option with lookback feature is the Russian option (perpetual American lookback option). Closed form price formulas of Russian options have been derived in several papers [Duffie and Harrison (1993); Shepp and Shiryaev (1993)]. Lim (1998) and Yu *et al.* (2001) examined the exercise boundaries of one-asset American lookback options. In this section, we would like to analyze the behaviors of the early exercise policies of two types of American quanto lookback options, whose exercised payoffs are defined in Eqs. (2.4a,b). To proceed with the analysis, we first state the linear complimentary formulation of the pricing models, then examine some monotonicity properties of the price functions and the exercise boundaries.

3.1 American maximum exchange rate quanto call

Let $V_M(S, F, \tau; F_{max})$ denote the value of an American maximum rate quanto call option in domestic currency, where τ is the time to expiry and F_{max} is the realized maximum exchange rate up to the current time. The linear complimentarity formulation for $V_M(S, F, \tau; F_{max})$ is given by

$$\begin{aligned}
&\frac{\partial V_M}{\partial \tau} - LV_M \geq 0, \quad V_M \geq F_{max} \max(S - K, 0), \\
&\left(\frac{\partial V_M}{\partial \tau} - LV_M\right)[V_M - F_{max} \max(S - K, 0)] = 0, \quad S > 0, 0 < F < F_{max}, \tau \in [0, T], \\
&\left.\frac{\partial V_M}{\partial F_{max}}\right|_{F_{max}=F} = 0 \quad \text{and} \quad V_M(S, F, 0; F_{max}) = F_{max} \max(S - K, 0),
\end{aligned} \tag{3.1}$$

where L is the differential operator defined by

$$\begin{aligned}
L = {} & \frac{\sigma_S^2}{2} S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma_S \sigma_F SF \frac{\partial^2}{\partial S \partial F} + \frac{\sigma_F^2}{2} F^2 \frac{\partial^2}{\partial F^2} \\
& + \delta_S^d S \frac{\partial}{\partial S} + (r_d - r_f) F \frac{\partial}{\partial F} - r_d.
\end{aligned} \tag{3.2}$$

In the continuation region, V_M satisfies

$$\frac{\partial V_M}{\partial \tau} - LV_M = 0 \qquad \text{and} \qquad V_M > F_{max} \max(S - K, 0); \tag{3.3a}$$

while in the exercise region, V_M satisfies

$$\frac{\partial V_M}{\partial \tau} - LV_M > 0 \qquad \text{and} \qquad V_M = F_{max} \max(S - K, 0). \tag{3.3b}$$

In the above linear complimentarity formulation, F_{max} appears apparently as a parameter. In the subsequent analysis, it is more convenient to use F_{max} as the numeraire and consider the monotonicity properties on the normalized price function

$$U_M(S, \xi, \tau) = V_M(S, F, \tau; F_{max})/F_{max}, \quad \text{where} \quad \xi = F/F_{max}. \tag{3.4}$$

We would like to explore some analytic behaviors of the critical stock price at which it is optimal to exercise the American maximum rate quanto lookback call option. We write the critical stock price as $S_M^*(\xi, \tau)$ with its dependence on ξ and τ.

Proposition 4

The normalized price function $U_M(S, \xi, \tau)$ satisfies the following monotonicity properties with respect to τ, ξ and the strike price K.

(a) $\dfrac{\partial U_M}{\partial \tau} \geq 0$

(b) $\dfrac{\partial U_M}{\partial \xi} \geq 0$

(c) $U_M(S, \xi, \tau; K_1) - U_M(S, \xi, \tau; K_2) \leq K_2 - K_1$ with $K_2 > K_1$.

Similar to other American call options, the continuation region and the exercise region correspond to $S < S_M^*(\xi, \tau)$ and $S \geq S_M^*(\xi, \tau)$, respectively. This would imply that if $U_M(S_1, \xi, \tau) = S_1 - K$, then $U_M(S_2, \xi, \tau) = S_2 - K$ for all $S_2 > S_1$. This property can be established using the inequality in part (c) and the linear homogeneity property of $U_M(S, \xi, \tau; K)$ with respect to S and K. Using the above monotonicity properties on U_M, we are able to obtain the following analytic properties on $S_M^*(\xi, \tau)$.

Theorem 5

Consider the optimal exercise boundary $S_M^*(\xi, \tau)$.

(a) At time close to expiry, $\tau \to 0^+$, we have

$$S_M^*(\xi, 0^+) = \begin{cases} \max\left(1, \dfrac{r_d}{r_d - \delta_S^d}\right) K & \text{if } r_d > \delta_S^d \\ \infty & \text{if } r_d \leq \delta_S^d \end{cases}. \tag{3.5}$$

(b) $S_M^*(\xi, \tau)$ is monotonically increasing with respect to τ and ξ.

Remark

From Theorem 5, we conclude that when $r_d \leq \delta_S^d$, it is never optimal to exercise the American maximum exchange rate quanto call prematurely. By virtue of the monotonicity property of the critical stock price on ξ, $S_M^*(F, \tau; F_{max})$ would increase with increasing F for fixed F_{max} and decrease with increasing F_{max} for fixed F.

We performed numerical calculations on the characterization of the exercise boundaries so as to the verify the results obtained in Theorem 5. Figure 1 shows the exercise boundaries of an American maximum exchange rate quanto call option at different times to expiry τ. The parameter values of the option model are $r_d = 0.05, r_f = 0.05, q = 0.02, \sigma_S = 0.2, \sigma_F = 0.2, \rho = 0.5, K = 1$, with $\dfrac{r_d}{r_d - \delta_S^d} K = 1.25$. The monotonicity properties on $S_M^*(\xi, \tau)$ with respect to ξ and τ are clearly revealed in Figure 1. The exercise region and the continuation region are on the right side and the left side of the exercise boundary, respectively. It is interesting to observe that $S_M^*(\xi, \tau)$ changes abruptly at some threshold level of ξ. When ξ increases beyond this τ-dependent threshold level, $S_M^*(\xi, \tau)$ increases quite substantially implying that the holder will wait for much significant increase in stock price in order to exercise the quanto lookback call option. In particular, when F becomes close to F_{max}, $S_M^*(\xi, \tau)$ becomes exceedingly large. This is reasonable since it is much likely that a higher value of F_{max} will be realized later so the holder should restrain from exercising the option prematurely.

The theoretical analysis of the monotonicity property of $V_M(S, F, \tau; \rho)$ with respect to the correlation coefficient ρ is not straightforward, due to the presence of ρ in both the covariance term $\rho\sigma_S\sigma_F SF \dfrac{\partial^2 V_M}{\partial S \partial F}$ and the drift term $\delta_S^d S \dfrac{\partial V_M}{\partial S}$. Since the drift term is expected to predominate over the covariance term and δ_S^d is a decreasing function of ρ, the option value $V_M(S, F, \tau; \rho)$ would be expected to be a decreasing function of ρ. Actually, the same monotonicity behavior on ρ is observed in other quanto call options [Kwok and Wong (2000)]. In all our wide range of numerical experiments that were performed to testify this monotonicity property, we observed that

$V_M(S,F,\tau;\rho)$ always appears to be a monotonically decreasing function of ρ. In Figure 2, we show the result of a typical calculation where $V_M(S,F,\tau;\rho)$ decreases monotonically with increasing ρ. The parameter values used in the calculation are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, T = 0.1$ and $K = S = F = F_{max} = 1$.

3.2 American joint quanto fixed strike lookback call option

Let $V_J(S,F,\tau;S_{max})$ denote the value of an American joint quanto fixed strike lookback call option in domestic currency, where S_{max} is the realized maximum value of the stock price up to the current time. The linear complimentarity formulation for $V_J(S,F,\tau;S_{max})$ is given by

$$\begin{aligned}
&\frac{\partial V_J}{\partial \tau} - LV_J \geq 0, \qquad V_J \geq \max(F,F_c)\max(S_{max}-K,0),\\
&\left(\frac{\partial V_J}{\partial \tau} - LV_J\right)[V_J - \max(F,F_c)\max(S_{max}-K,0)] = 0, \quad F>0, 0<S<S_{max}, \tau\in[0,T],\\
&\left.\frac{\partial V_J}{\partial S_{max}}\right|_{S_{max}=S} = 0 \quad \text{and} \quad V_J(S,F,0;S_{max}) = \max(F,F_c)\max(S_{max}-K,0), \qquad (3.6)
\end{aligned}$$

where L is the differential operator defined in Eq. (3.2). In the continuation region, V_J satisfies

$$\frac{\partial V_J}{\partial \tau} - LV_J = 0 \quad \text{and} \quad V_J > \max(F,F_c)\max(S_{max}-K,0); \qquad (3.7a)$$

while in the exercise region, V_J satisfies

$$\frac{\partial V_J}{\partial \tau} - LV_J > 0 \quad \text{and} \quad V_J = \max(F,F_c)\max(S_{max}-K,0). \qquad (3.7b)$$

Gerber and Shiu (1996) showed that the exercise boundary of an American option on the maximum of two stock prices with zero strike consists of two branches. When the two stock prices are close in value, the holder of this American option should delay premature exercise. This is because the advantage of choosing the maximum of the two stock prices is not distinctive when the stock prices are about the same value. Only when either one of the stock prices is significantly higher than the other should the American option holder chooses to exercise. In this scenario, the chance of regret of premature exercise would be low.

Due to the presence of the factor $\max(F,F_c)$ in the payoff function, the exercise boundary of an American joint quanto lookback call would be expected to consist of two branches: $F^*_{up}(S,\tau;S_{max})$ and $F^*_{low}(S,\tau;S_{max})$. Obviously, early exercise is advantageous only when $S_{max} > K$, that is, the option is currently in-the-money. When $S_{max} > K$ but the value of F is close to the predetermined constant F_c, the holder should delay premature exercise since the advantage of taking the maximum of F and F_c is not significant. The chance of regret of early exercise is low only when F is sufficiently above F_c or below F_c. In the F-τ plane, conditional on $S_{max} > K$, the continuation region is bounded by the two branches of the exercise boundary: $F^*_{up}(S,\tau;S_{max})$ and $F^*_{low}(S,\tau;S_{max})$. When $F \geq F^*_{up}$ or $F \leq F^*_{low}$, it becomes optimal to exercise the American joint quanto lookback call. Therefore, one part of the exercise region is to the right side of the branch $F^*_{up}(S,\tau;S_{max})$ and the other part is to the left of $F^*_{low}(S,\tau;S_{max})$.

We performed numerical calculations to compute the early exercise boundary of the American joint quanto lookback call. In Figure 3, we show the plots of the two branches of the exercise boundary corresponding to different pairs of values of S and S_{max}. The parameter values used in the calculations are $r_d = 0.05, r_f = 0.05, q = 0.02, \sigma_S = 0.2, \sigma_F = 0.2, \rho = 0.5, F_c = 1, K = 1$ and $T = 0.1$. The two branches $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ both originate from $F = F_c$ at $\tau \to 0^+$. We observe that $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are, respectively, monotonically increasing and decreasing with respect to τ. When τ reaches some threshold value, it is interesting to see that F^*_{up} increases and F^*_{low} decreases quite drastically with increasing τ. Besides, for a fixed value of τ, F^*_{up} is monotonically decreasing with respect to S_{max} (S is fixed) but monotonically increasing with respect to S (S_{max} is fixed). The corresponding monotonicity properties on F^*_{low} are reverse to those on F^*_{up}. These monotonicity properties can be explained by intuitive arguments that look into the chance of regret of premature exercise. The chance of regret decreases with increasing value of S_{max} (option being deeper in-the-money) and decreasing value of S (less chance to realize a new maximum value of the stock price in the future).

In Figure 4, we plot the option value of the American joint quanto lookback call at different times to expiry τ. We choose $S = 1$ and $S_{max} = 1.33$, and other parameters of the option model are identical to those used in Figure 3. The intrinsic value, $\max(F, F_c) \max(S_{max} - K, 0)$, of the lookback call is represented by the dotted horizontal line and inclined line. It is observed that each option value curve intersects tangentially the intrinsic value lines at F^*_{up} above F_c and at F^*_{low} below F_c. Also, the option value is seen to be monotonically increasing with respect to τ.

Some of the properties of the exercise policy and exercise boundary of the American joint quanto lookback call option are stated in Theorem 6.

Theorem 6

The exercise boundary of the American joint quanto fixed strike lookback call option in the F-τ plane consists of two branches: $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$. For fixed values of τ, S and S_{max}, conditional on $S_{max} > K$, the option should be optimally exercised when $F \geq F^*_{up}$ or $F \leq F^*_{low}$. The continuation region lies within $F^*_{low}(S, \tau; S_{max})$ and $F^*_{up}(S, \tau; S_{max})$. The two branches of the exercise boundary intersect at $F = F_c$ at $\tau \to 0^+$. At time close to expiry, conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate F other than F_c.

4. Conclusion

The analytic price formulas of two types of European quanto lookback options have been derived. The analytic tractability of these two-state lookback option models has been extended via the use of a succinct analytic representation of the density function of the joint process of the extreme value and terminal value of the exchange rate and stock price. The price formulas help provide the financial interpretation of the contributing factors to the value of the European quanto lookback option.

We have also analyzed the characterization of the exercise boundaries and pricing behaviors of these two types of quanto lookback options with the early exercise privilege. For the American maximum exchange rate quanto call, the critical stock price $S^*_M(F, \tau; F_{max})$ at which it is optimal to exercise the option is seen to be monotonically increasing with respect to time to expiry τ and exchange rate F (for fixed realized maximum exchange rate F_{max}). We show that it is never optimal to exercise the maximum exchange rate quanto call if the effective dividend yield of the foreign stock in domestic currency world is non-positive. Also, when F comes close to F_{max}, it becomes much less likely to exercise the option prematurely. For the American joint quanto fixed strike lookback call, the exercise boundary consists of two branches. Conditional on the option being in-the-money (current realized maximum stock price S_{max} is higher than the strike price K), it is optimal to exercise the option only when the exchange rate F is either sufficiently above or below the predetermined constant exchange rate F_c. At time right before expiry, it is optimal to exercise the American joint quanto lookback call at any level of exchange rate F other than F_c. These results add new insights into the understanding of the characterization of the early exercise policies of the general class of multi-asset American options.

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Appendix

Proof of Theorem 1

The two functions $g_{max}(x, y, T; m)$ and $G_{max}(x, m, T)$ are the density function and distribution function defined by

$$
\begin{aligned}
g_{max}(x, y, T; m)\ dxdy &= P(X \in dx, Y \in dy, M \leq m),\\
G_{max}(x, m, T)\ dx &= P(X \in dx, M \leq m).
\end{aligned}
$$

The governing equation for $g_{max}(x, y, t; m)$ is the two-dimensional forward Fokker-Planck equation, where

$$
\frac{\partial g_{max}}{\partial t} = -\mu_X \frac{\partial g_{max}}{\partial x} - \mu_Y \frac{\partial g_{max}}{\partial y} + \frac{1}{2}\frac{\partial^2 g_{max}}{\partial x^2} + \rho \frac{\partial^2 g_{max}}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 g_{max}}{\partial y^2}
$$
$$
-\infty < x < \infty, -\infty < y < m, t > 0,
$$

with auxiliary conditions

$$
g_{max}(x, y, 0; m) = \delta(x)\delta(y) \quad \text{and} \quad g_{max}(x, m, t; m) = 0.
$$

Here, $y = m$ is an absorbing barrier for the random process Y_t. By solving the above Fokker-Planck equation, we obtain the solution to $g_{max}(x, y, T; m)$ as given in Eq. (2.11b).

Proof of Theorem 2

From Eqs. (2.9), (2.10) and (2.11a-c), we obtain

$$
\begin{aligned}
V_{max} = e^{-r_d T} F e^{\sigma_F M_0} &\int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_0^{M_0} (S e^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, m, T)\ dmdx\\
+\ e^{-r_d T} F &\int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{M_0}^{\infty} e^{\sigma_F m}(S e^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, m, T)\ dmdx. \qquad (A2.1)
\end{aligned}
$$

By performing the inner integration with respect to m, the first integral can be expressed as

$$
\begin{aligned}
I_1 = e^{-r_d T} F e^{\sigma_F M_0} &\int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{M_0} (S e^{\sigma_S x} - K)\\
&[\phi_2(\widetilde{x}, \widetilde{y}, T; \rho) - e^{2\mu_Y M_0}\phi_2(\widetilde{x} - 2\rho M_0, \widetilde{y} - 2M_0, T; \rho)]\ dydx. \qquad (A2.2)
\end{aligned}
$$

For the second term in Eq. (A2.1), we consider the second term in $\dfrac{\partial G_{max}}{\partial m}$ and apply parts integration to obtain

$$
\begin{aligned}
&-\int_{M_0}^{\infty} e^{\sigma_F m} \frac{\partial}{\partial m}\left[\int_{-\infty}^{m} e^{2\mu_Y m}\phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dy\right] dm\\
&= e^{\sigma_F M_0} \int_{-\infty}^{M_0} e^{2\mu_Y M_0}\phi_2(\widetilde{x} - 2\rho M_0, \widetilde{y} - 2M_0, T; \rho)\ dy\\
&\quad +\ \sigma_F \int_{M_0}^{\infty} e^{(2\mu_Y + \sigma_F)m} \int_{-\infty}^{m} \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dydm. \qquad (A2.3)
\end{aligned}
$$

Note that the second term in I_1 [see Eq. (A2.2)] cancels with the double integral arising from the first term in Eq. (A2.3). By observing this cancellation, Eq. (A2.1) can be expressed as

$$\begin{aligned} V_{max} = e^{-r_d T} F \Bigg[& e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{M_0} (Se^{\sigma_S x} - K)\phi_2(\widetilde{x}, \widetilde{y}, T; \rho)\, dydx \\ & + \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{M_0}^{\infty} e^{\sigma_F m}(Se^{\sigma_S x} - K)\phi_2(\widetilde{x}, \widetilde{m}, T; \rho)\, dmdx \\ & + \sigma_F \int_{M_0}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{m} e^{(\sigma_F + 2\mu_Y)m}(Se^{\sigma_S x} - K) \\ & \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\, dydxdm \Bigg], \end{aligned} \tag{A2.4}$$

where $\widetilde{m} = m - \mu_Y T$. Here, the first two integrals are expressible in terms of $N_2(\cdot\,,\cdot\,;\rho)$ while the last integral can be simplified to become a single integral with the integrand involving $N_2(\cdot\,,\cdot\,;\rho)$.

Proof of the zero derivative condition at $F = F_{max}^{[T_0,0]}$

By differentiating V_{max} in Eq. (A2.1) with respect to M_0, we obtain

$$\begin{aligned} \frac{\partial V_{max}}{\partial M_0} = e^{-r_d T} F \Bigg[& \sigma_F e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{0}^{M_0} (Se^{\sigma_S x} - K)\frac{\partial G_{max}}{\partial m}(x, m, T)\, dmdx \\ & + e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (Se^{\sigma_S x} - K)\frac{\partial G_{max}}{\partial m}(x, M_0, T)\, dx \\ & - \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F M_0}(Se^{\sigma_S x} - K)\frac{\partial G_{max}}{\partial m}(x, M_0, T)\, dx \Bigg]. \end{aligned}$$

The second and the third terms cancel with each other, and the first term becomes zero when M_0 is set equal to zero. Hence, we obtain

$$\left.\frac{\partial V_{max}}{\partial M_0}\right|_{M_0=0} = 0.$$

Proof of Theorem 3

We consider the following two separate cases:

1. $S_{max}^{[T_0,0]} < K$

   By observing that $U_0 < \dfrac{1}{\sigma_S}\ln\dfrac{K}{S}$ and the first term in $g_{joint}(x, y, T; u)$ is independent of u, we transform V_{joint} in Eq. (2.18) into the following form

$$
\begin{aligned}
V_{joint} = e^{-r_d T} \Bigg\{ & F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (Se^{\sigma_S u} - K)\phi_2(\widetilde{u}, \widetilde{y}, T; \rho)\, dudy \\
& - F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (Se^{\sigma_S u} - K) \\
& \quad \frac{\partial}{\partial u}\left[\int_{-\infty}^{u} e^{2\mu_X u}\phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\, dx\right] dudy \\
& + F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K)\phi_2(\widetilde{u}, \widetilde{y}, T; \rho)\, dudy \\
& - F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K) \\
& \quad \frac{\partial}{\partial u}\left[\int_{-\infty}^{u} e^{2\mu_X u}\phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\, dx\right] dudy \Bigg\},
\end{aligned}
$$

where $\widetilde{u} = u - \mu_X T$. The first and third integrals can be expressed in terms of $N_2(\cdot\,, \cdot\,; \rho)$ in a straightforward manner. By applying parts integration, the second and the fourth integrals can be expressed as

$$
\begin{aligned}
\text{second integral } &= e^{-r_d T} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\
& \left[F_c S \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{-\infty}^{u} \phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\, dxdy \right] du, \\
\text{fourth integral} &= e^{-r_d T} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\
& \left[FS \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{-\infty}^{u} e^{\sigma_F y}\phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\, dxdy \right] du.
\end{aligned}
$$

Both of the above two integrals can be expressed as a single integral with integrand involving $N_2(\cdot\,, \cdot\,; \rho)$.

2. $S_{max}^{[T_0,0]} \geq K$

The first and third integrals in Eq. (2.19) can be expressed as

$$
e^{-r_d T} F_c (S_{max}^{[T_0,0]} - K) \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{-\infty}^{U_0} \phi_2(x, y, T; \rho)\, dxdy
$$

and

$$
e^{-r_d T} F (S_{max}^{[T_0,0]} - K) \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{-\infty}^{U_0} e^{\sigma_F y}\phi_2(x, y, T; \rho)\, dxdy,
$$

respectively. The second and the fourth integrals are similar to those in Eq. (2.18) except that the lower limit becomes U_0 instead of $\dfrac{1}{\sigma_S}\ln\dfrac{K}{S}$.

Proof of Proposition 4

(a) For any American options, the value of the longer-lived one is always worth at least that of its shorter-lived counterpart, so $\dfrac{\partial U_M}{\partial \tau} = \dfrac{1}{F_{max}}\dfrac{\partial V_M}{\partial \tau} \geq 0$.

(b) For a given value of F_{max}, $V_M(S,F,\tau)$ is a non-decreasing function of F since a higher value of F would mean at least the same or a higher value of $F_{max}^{[T_0,T]}$ to be realized at expiry compared to the counterpart with a lower value of F. We then have $\dfrac{\partial U_M}{\partial \xi} = \dfrac{\partial V_M}{\partial F} \geq 0$.

(c) Consider two strike prices K_1 and K_2 with $K_2 > K_1$, and define $W_i(S,\xi,\tau) = U_M(S,\xi,\tau;K_i) + K_i, i = 1,2$. Substituting $W_i(S,\xi,\tau)$ into the linear complimentarity formulation (3.1), we obtain

$$\frac{\partial W_i}{\partial \tau} - \widehat{L}W_i \geq r_d K_i, \qquad W_i \geq \max(S, K_i),$$
$$\left(\frac{\partial W_i}{\partial \tau} - \widehat{L}W - r_d K_i\right)[W_i - \max(S,K_i)] = 0,$$
$$W_i - \left.\frac{\partial W_i}{\partial \xi}\right|_{\xi=1} = K_i \quad \text{and} \quad W_i(S,\xi,0) = \max(S,K_i),$$

where

$$\widehat{L} = \frac{\sigma_S^2}{2}S^2\frac{\partial^2}{\partial S^2} + \rho\sigma_S\sigma_F S\xi\frac{\partial^2}{\partial S\partial\xi} + \frac{\sigma_F^2}{2}\xi^2\frac{\partial^2}{\partial \xi^2}$$
$$+ \delta_S^d S\frac{\partial}{\partial S} + (r_d - r_f)\xi\frac{\partial}{\partial \xi} - r_d.$$

Since $K_2 > K_1$, by virtue of the comparison principle in partial differential equation theory, it is obvious that $W_2(S,\xi,\tau) > W_1(S,\xi,\tau)$; and hence the result.

Proof of Theorem 5

The monotonicity property: $\dfrac{\partial U_M}{\partial \tau} > 0$ is maintained in the continuation region even when $\tau \to 0^+$. First, it is obvious that $S_M^*(\xi,0^+) \geq K$. For $S \in (K, S_M^*(\xi,0^+))$, we have $U_M(S,\xi,0^+) = S - K$. Since $U_M(S,\xi,0^+)$ should satisfy $\dfrac{\partial U_M}{\partial \tau} = \widehat{L}U_M$, we obtain

$$\left.\frac{\partial U_M}{\partial \tau}\right|_{\tau=0} = \delta_S^d S - r_d(S-K) = r_d K - (r_d - \delta_S^d)S.$$

For $r_d > \delta_S^d$, the condition: $\left.\dfrac{\partial U_M}{\partial \tau}\right|_{\tau=0} > 0$ is satisfied only for $S < \dfrac{r_d}{r_d - \delta_S^d}K$. On the other hand, when $r_d \leq \delta_S^d$, $\left.\dfrac{\partial U_M}{\partial \tau}\right|_{\tau=0} > 0$ always holds true. We then conclude that

$$S_M^*(\xi,0^+) = \begin{cases} \max\left(1, \dfrac{r_d}{r_d - \delta_S^d}\right)K & \text{if } r_d > \delta_S^d \\ \infty & \text{if } r_d \leq \delta_S^d \end{cases}.$$

The above result agrees with the usual result for critical asset price close to expiry for American call options when we visualize $r_d - \delta_S^d$ as the effective dividend yield of the foreign stock in the domestic currency world.

To show the monotonicity property of $S_M^*(\xi, \tau)$ with respect to τ, we let $\tau_2 > \tau_1$ and consider the evaluation of $U_M(S, \xi, \tau)$ at stock price level $S = S_M^*(\xi, \tau_1)$ and at two times τ_1 and τ_2. By virtue of the monotonicity property of U_M on τ, we have

$$U_M(S_M^*(\xi, \tau_1), \tau_2) > U_M(S_M^*(S, \tau_1), \tau_1) = S_M^*(\xi, \tau_1) - K.$$

This implies that the American option remains in the continuation region when $S = S_M^*(\xi, \tau_1)$ and $\tau = \tau_2$. Since the exercise region is on the right side of the continuation region, we deduce that

$$S_M^*(\xi, \tau_2) > S_M^*(\xi, \tau_1), \qquad \tau_2 > \tau_1.$$

The monotonicity property of $S_M^*(\xi, \tau)$ with respect to ξ can be established by using the monotonicity property of U_M on ξ and following a similar argument as above.

Proof of Theorem 6

At $F = F_c$ and $\tau > 0$, conditional on $S_{max} > K$, the option should remain alive. If otherwise, the option value is equal to the exercised payoff. Substituting $V_J = \max(F, F_c)(S_{max} - K)$ into Eq. (3.7b), we observe that

$$\begin{aligned}\frac{\partial V_J}{\partial \tau} - LV_J = -\Bigg[&\frac{\sigma_F^2}{2} F^2 \delta(F - F_c)(S_{max} - K) + (r_d - r_f)F(S_{max} - K)\mathbf{1}_{\{F > F_c\}} \\ &- r_d \max(F, F_c)(S_{max} - K)\Bigg] \longrightarrow -\infty \text{ when } F = F_c,\end{aligned}$$

where $\delta(x)$ and $\mathbf{1}_A$ are the delta function and indicator function, respectively. Since the condition: $\dfrac{\partial V_J}{\partial \tau} - LV_J \geq 0$ is not satisfied, the option should not be optimally exercised at $F = F_c$ and $\tau > 0$. The whole vertical line $F = F_c$ in the F-τ plane lies in the continuation region.

Next, we would like to show that the exercise regions contain the two horizontal line segments: $\{\tau = 0, F < F_c\}$ and $\{\tau = 0, F > F_c\}$ in the F-τ plane. Assume the contrary, suppose there exists a finite interval $(F_{low}^*(S, 0^+), F_{up}^*(S, 0^+))$ at $\tau \to 0^+$ that lies completely within the continuation region. Let $F \in (F_{low}^*(S, 0^+), F_{up}^*(S, 0^+))$; by continuity, the option value evaluated at F and $\tau \to 0^+$ is $\max(F, F_c)(S_{max} - K)$. Substituting this option value into Eq. (3.7a), we then have

$$\left.\frac{\partial V_J}{\partial \tau}\right|_{\tau=0} = \begin{cases} -r_f F(S_{max} - K) & \text{for } F > F_c \\ -r_d F(S_{max} - K) & \text{for } F < F_c \end{cases}.$$

In both cases, $\left.\dfrac{\partial V_J}{\partial \tau}\right|_{\tau=0} < 0$, which is in contradiction to the property: $\left.\dfrac{\partial V_J}{\partial \tau}\right|_{\tau=0} \geq 0$. This would then imply the non-existence of such finite interval. Hence, at time close to expiry and conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate F other than F_c.

In the F-τ plane, the vertical line $F = F_c$ is in the continuation region while the two horizontal line segments: $\{\tau = 0, F < F_c\}$ and $\{\tau = 0, F > F_c\}$ are in the exercise regions. We then deduce that for a fixed value of τ, there exist some critical values F^*_{up} and $F^*_{low}(F^*_{up} > F_c$ and $F^*_{low} < F_c)$ such that the option should be optionally exercised when $F \geq F^*_{up}$ or $F \leq F^*_{low}$ (see Figure 4). Due to the monotonic increasing property of the option value with respect to τ, it can be shown that $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are unique. In other words, the exercise boundary consists of exactly one branch $F^*_{up}(S, \tau; S_{max})$ that lies completely to the right of the vertical line $F = F_c$ and another unique branch $F^*_{low}(S, \tau; S_{max})$ to the left of $F = F_c$. The two branches $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ intersect at $F = F_c$ when $\tau \to 0^+$. Further, $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are, respectively, monotonically increasing and decreasing with respect to τ.

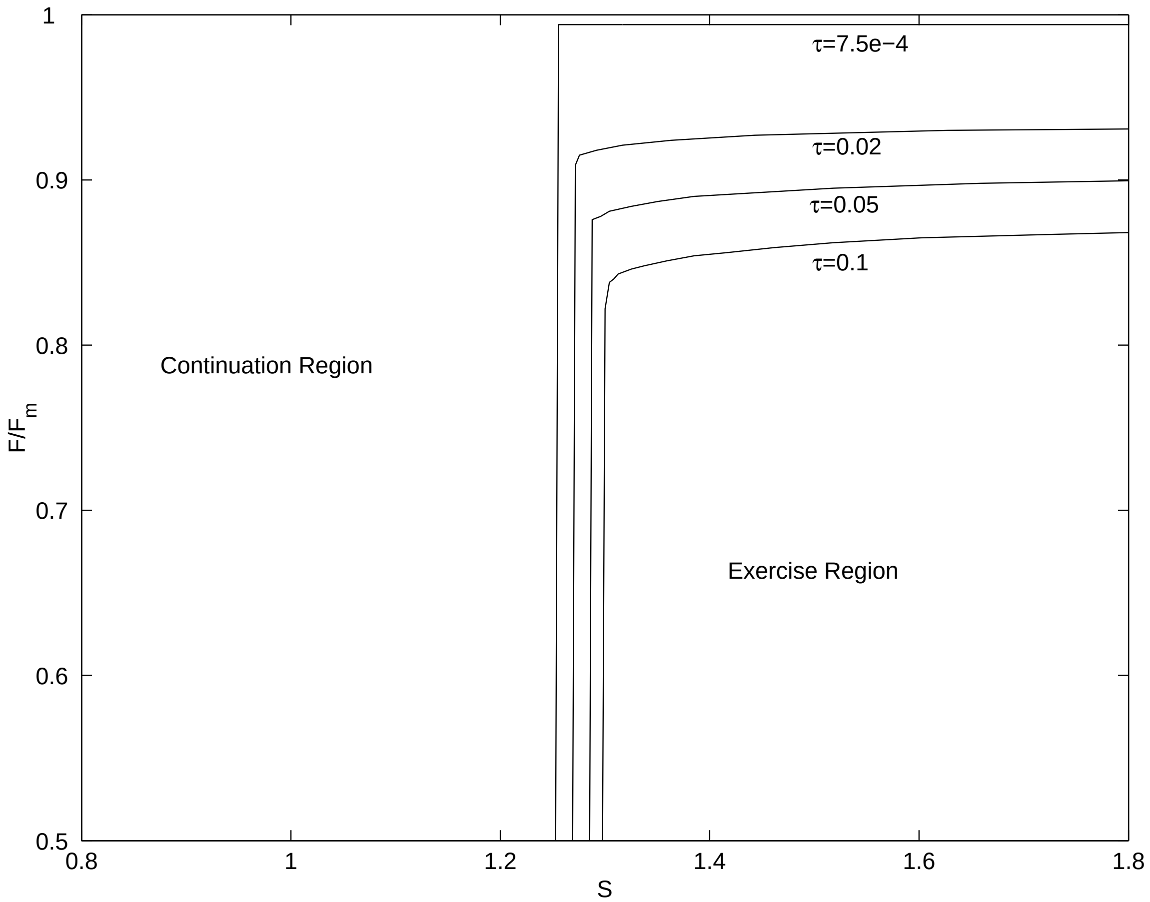

Figure 1 The exercise boundaries of an American maximum exchange rate quanto call option at different times to expiry τ are plotted. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5$ and $K = 1$. The exercise region and the continuation region are on the right and left side of the exercise boundary respectively.

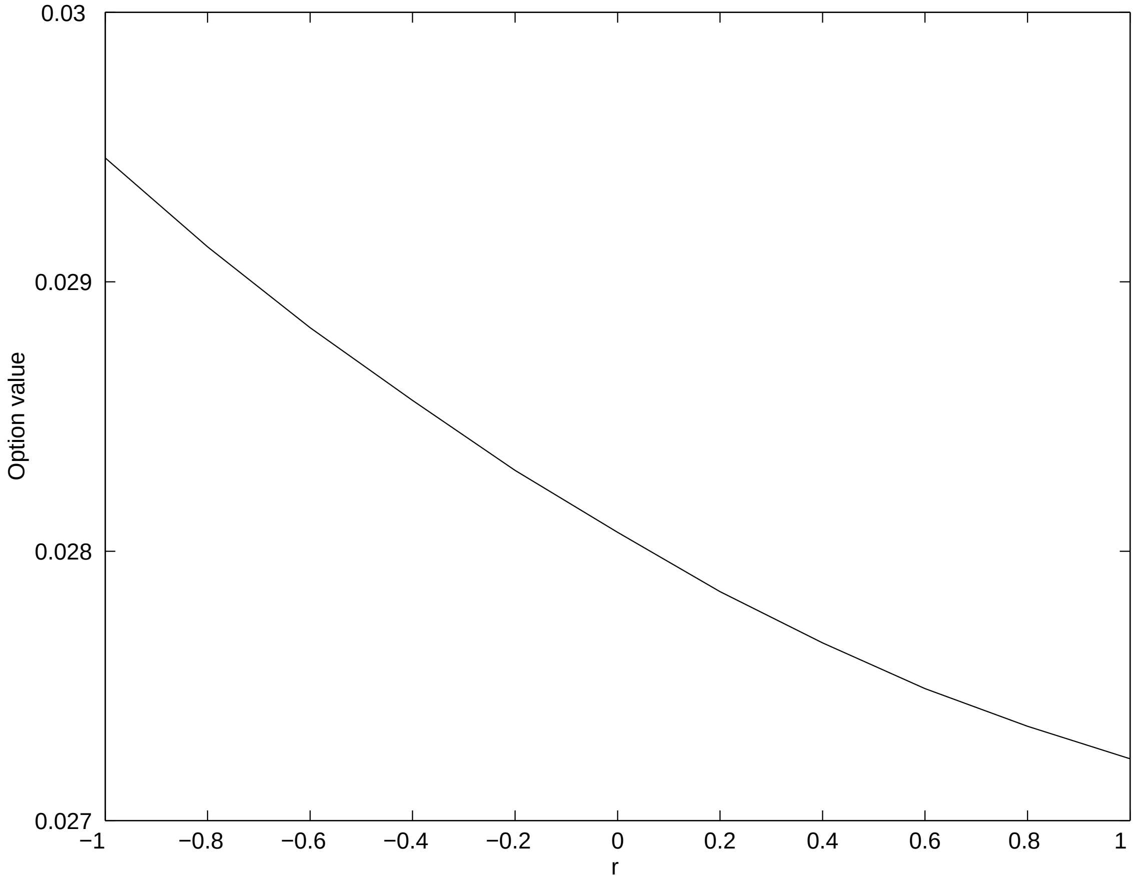

Figure 2 The value of an American maximum exchange rate quanto call option is plotted against the correlation coefficient ρ. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, T = 0.1$ and $K = S = F = F_{max} = 1$.

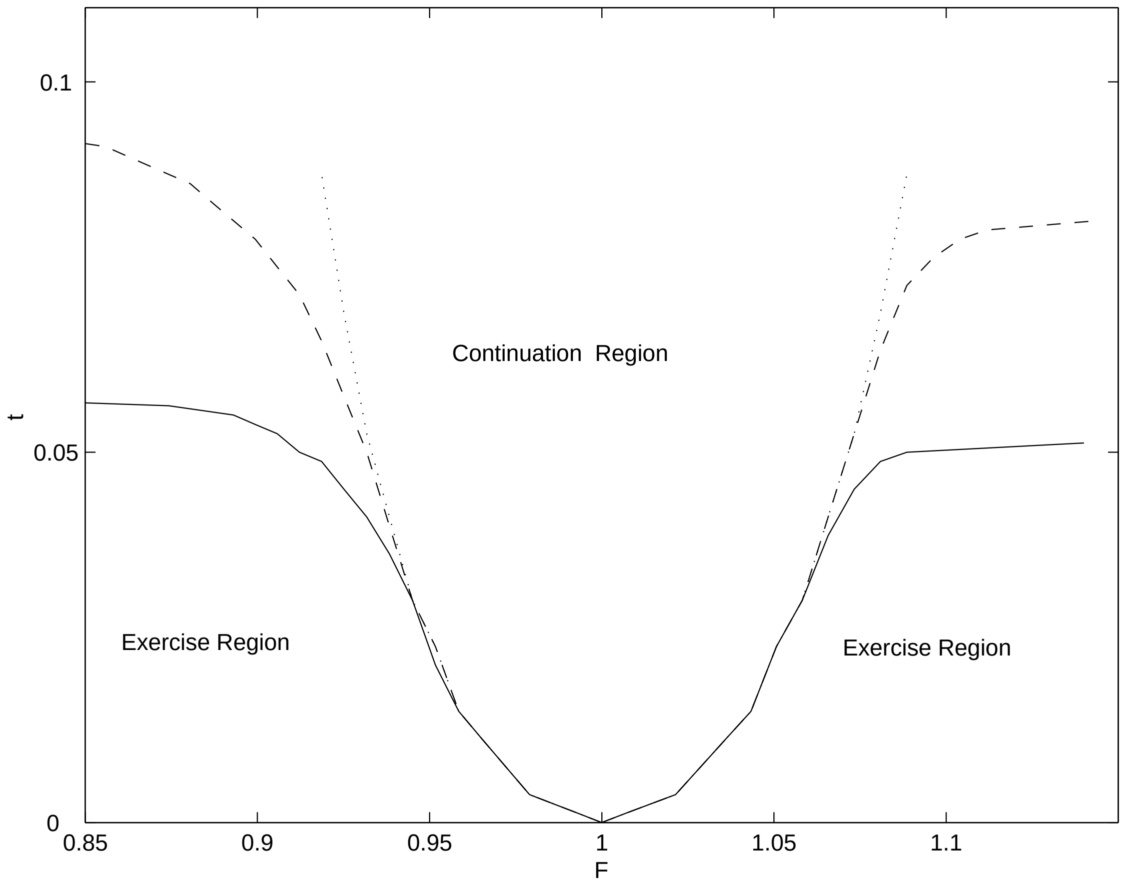

Figure 3 The exercise boundaries of an American joint quanto fixed strike lookback call option at different pairs of values of S and S_{max} are plotted. The parameters in the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5, K = F_c = 1$ and $T = 0.1$. The solid curve corresponds to $S = 1.17, S_{max} = 1.30$; the dashed curve corresponds to $S = 1.17, S_{max} = 1.33$; and the dotted curve corresponds to $S = 1.00, S_{max} = 1.33$. The exercise boundary consists of two branches with the continuation region lying in between.

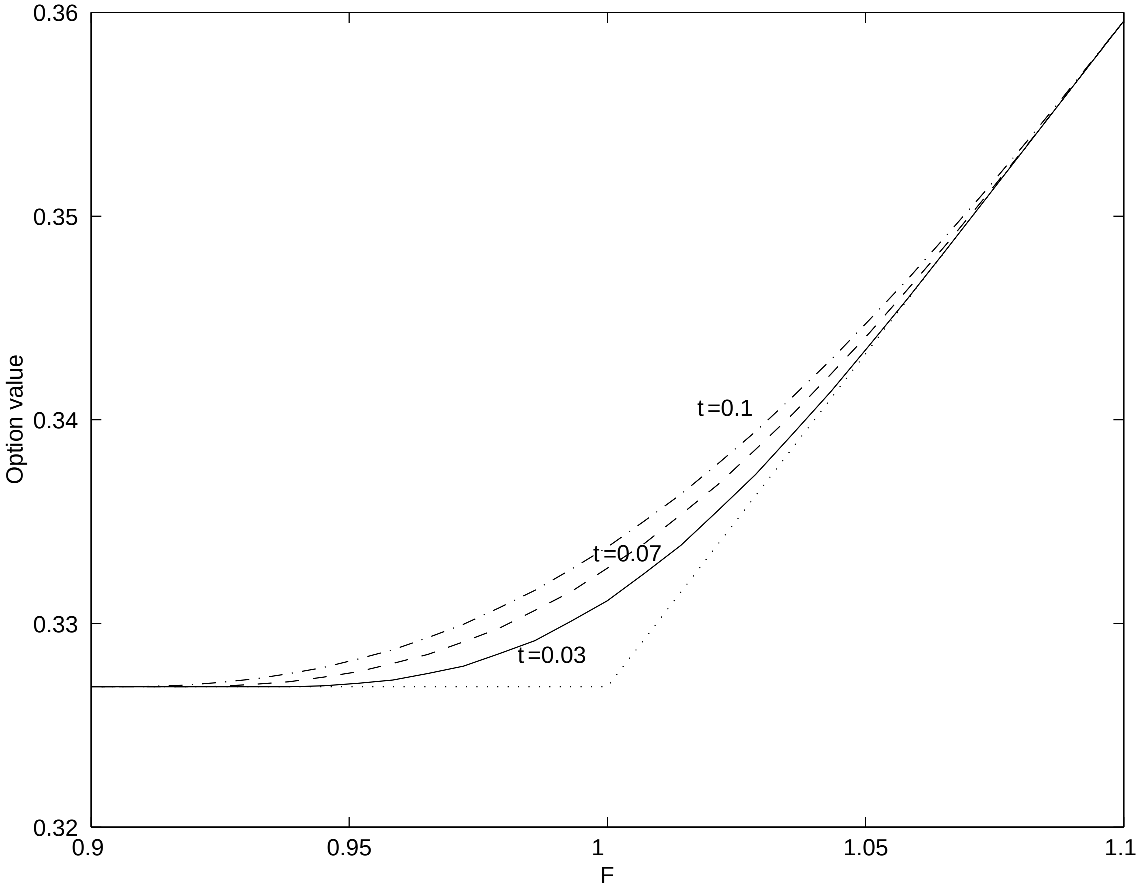

Figure 4 The value of an American joint quanto fixed strike lookback call option is plotted against the exchange rate F at different times to expiry τ. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5, K = 1, F_c = S = 1$ and $S_{max} = 1.33$. Each option value curve cuts tangentially the intrinsic value lines (shown as dotted lines) at two critical exchange rates $F^*_{low}(S, \tau; S_{max})$ and $F^*_{up}(S, \tau; S_{max})$.