Sub-replication and Replenishing Premium: Efficient Pricing of Multi-state Lookbacks

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Abstract

The most common approach to derive the price formula of a financial derivative is the calculation of the discounted expectation of the terminal payoff under the risk neutral measure. However, the use of such approach to derive the price formulas of multi-state lookback options may lead to insurmountable complexity in the analytic procedures. In this paper, we illustrate the use of an alternative approach that significantly simplifies the analytic derivation of multi-state lookback option price formulas. The new approach involves the choice of a sub-replicating portfolio and the adoption of the corresponding replenishing strategy to achieve the subsequent full replication of the derivative. Our work demonstrates the elegant use of financial intuition that greatly facilitates the analytic tractability in pricing exotic derivatives.

1. Introduction

Lookback options provide the opportunity for the holders to realize attractive gains in the event of substantial price movement of the underlying assets during the life of the option. To capture the price volatility of an asset, an investor may be interested to purchase a lookback option on the spread between the maximum and minimum prices of the underlying asset over a given time period. This option has come to be known as the *lookback spread option*. Also, one may structure lookback options on two underlying assets. The *semi-double lookback options* are options whose terminal payoff depends on the extreme value of one asset price and the terminal value of another asset price. If the terminal payoff of a lookback option depends on the extreme values of both asset prices, then the option is called a *full double lookback option*. All these types of lookback options can be collectively called *two-state lookback options* (He *et al.*, 1998). More exotic forms of lookback payoffs, like the hot dog option, bounded cliquet lookback, etc., are discussed in Babsiri and Noel's paper (1998).

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The pricing of lookback options poses interesting mathematical challenges. The analytic price formulas for one-asset lookback options have been systematically derived by Goldman *et al.* (1979), and Conze and Viswanathan (1991). For two-state lookback options, the analytic expressions of the joint probability density functions of the extreme values and terminal values of the prices of the underlying assets have been obtained by He *et al.* (1998) and Babsiri and Noel (1998). These probability density functions are used in the valuation of the lookback option prices via numerical integration of the discounted expectation integrals or Monte Carlo simulation.

A more careful examination of He *et al.*'s formulation reviews that their computational procedures involve double numerical differentiation of the density function followed by double numerical integration over infinite and/or semi-infinite intervals. It is well known that numerical differentiation is a highly unstable procedure and numerical integration over an infinite interval commonly faces with difficulties of treating the tailed region. In this paper, we illustrate how to obtain the price formulas for European style multi-state lookback options where the final analytic forms involve only single integration of a probability distribution function over a finite interval. Correspondingly, the complexity of numerical valuation of these price formulas is significantly reduced. Further, our methodology is not limited to pricing models where lognormal processes of the asset prices are assumed. The resulted simplicity of the price formulas stems from an elegant financial intuition. Instead of following the usual approach of evaluating the discounted risk neutral expectation of the terminal payoff, we choose a *sub-replicating portfolio* for the lookback option, then followed by the adoption of the corresponding *replenishing strategy* to achieve the full replication of the option.

The paper is organized as follows. In Section 2, we discuss the concepts of sub-replication and replenishing premium and illustrate the use of the technique to the pricing of European vanilla options. We then apply the methodology to derive the price formulas of one-asset European floating strike and fixed strike lookback options (the monitoring process for the extreme values can be continuous or discrete). The analogy between the replenishing premium and the strike bonus premium (Garman, 1992) is highlighted. In Section 3, we derive the price formulas of the one-asset and two-asset lookback spread options. The succinct representation of the price formula naturally reveals the financial intuition behind the derivation procedure. Our pricing methodology is applied further to the pricing of options on the extreme value of one asset and the terminal values of several assets in Section 4. The paper is ended with conclusive remarks in the last section. In the Appendix, we list different probability distribution functions (under the assumption of lognormal process for the asset prices) that occur in the price formulas of various lookback options derived in the paper.

2. Concepts of sub-replication and replenishing premium

The innovative concept of *riskless hedging* initiates the development of the option pricing theory. Black and Scholes (1973) showed that the risk of an option can be hedged by combining the option with an appropriate amount of the underlying asset to form a riskless portfolio. In order to avoid arbitrage, the riskless portfolio should earn the riskless interest rate. Alternatively, Merton (1973) showed that the option can be replicated by a portfolio of the underlying asset and the riskless bond. Assuming frictionless market and no premature termination of the option contract, suppose the option's payoff matches with that of the replicating portfolio at maturity, then the value of the option is equal to the value of the replicating portfolio at all times throughout the life of the option. If every derivative can be replicated by a portfolio of the fundamental assets in the market, then the market is said to be complete.

From the theory of financial economics, one can show that the condition of no arbitrage is equivalent to the existence of an equivalent martingale measure. Further, if the market is complete (all contingent claims can be replicated), then the equivalent martingale measure is unique. The above statements are the essence of the *Fundamental Theorem of Asset Pricing*. The Theorem leads to the identification of a new probability measure in the pricing of contingent claims, which is commonly called the *risk neutral probability*. It can be shown that the replication based price of any contingent claim can be obtained by calculating the discounted expected value of its terminal payoff under the risk neutral probability (Harrison and Kreps, 1979). This has come to be known as the risk neutral pricing. The term risk neutrality is used since all assets in the market offer the same return under this probability, so an investor who is neutral to risk and faces with this probability would be indifferent among various assets. The concepts of *replicable contingent claims, absence of arbitrage* and *risk neutrality* form the cornerstones of the modern option pricing theory.

In the literature, the price formulas of lookback options were derived based on the approach of calculating the discounted risk neutral expectation of the terminal payoff. In the coming subsections, we illustrate a new approach of derivative pricing through a careful examination of the pricing of the vanilla options and the one-asset floating strike and fixed strike lookback options. We then demonstrate the robustness of the new approach through the pricing of various types of multi-state lookback options in Sections 3 and 4. In this paper, the resulting price formulas are in general expressed as the sum of the prices of one or several elementary derivatives plus an integral with a probability distribution function as the integrand. For common types of asset price process, like the lognormal process, the analytic representations of the probability distribution functions are readily available in the literature. Besides, the integration limits in these integrals are finite values. Therefore, the numerical evaluation of these price formulas can be performed in a straightforward manner.

2.1 New perspective on the pricing of vanilla options

Consider a European call option with the strike price K, whose terminal payoff is given by $\max(S_T - K, 0)$. Here, S_T denotes the asset price at option's maturity T. From the payoff structure of the call, it is intuitive to compare the call with a portfolio which consists of long holding of one unit of the underlying asset and short selling of a riskless bond with par value K and same maturity as that of the option. The terminal payoff of the above portfolio is $S_T - K$, so the portfolio gives only a partial replication of the terminal payoff of the European call. This is because the portfolio and the call have the same terminal payoff only when the call expires in-the-money or at-the-money, correspondingly to $S_T \geq K$. The terminal value of the portfolio falls below that of the call option when $S_T < K$, that is, the call expires out-of-the-money. A partial replicating portfolio whose terminal value always stays equal or below the terminal value of the derivative to be replicated is said to be a *sub-replicating portfolio*. We normally choose a sub-replicating portfolio whose value is readily obtainable. The pricing of the call option then amounts to the determination of the additional premium for acquiring extra assets on top of the sub-replication that are required to achieve the full replication of the call. This additional premium is termed the *replenishing premium.*

The loss incurred to the writer of the call at maturity when the sub-replicating portfolio is employed to hedge the option's risk is given by the difference in the terminal payoffs of the call and the sub-replicating portfolio. This difference equals $K - S_T$ if $S_T < K$, and zero if otherwise. The writer is required to use additional assets to protect against the above loss scenario. In this case, we observe that the instrument required to replenish the mis-replication is simply the put option with strike K and same maturity date T. Let t denote the current time and write $\tau = T - t$. This comes no surprise since this is just the manifestation of the put-call parity relation. The replenishing premium is the value of the put.

For the purpose of enhancing analytic tractability in the derivation procedure, it is preferable that we write the replenishing premium in an integral form that involves the probability distribution function rather than the probability density function. Consider

$$\begin{aligned} \text{put value} &= e^{-r\tau} E\left[(K - S_T)\mathbf{1}_{\{S_T \leq K\}}\right] \\ &= e^{-r\tau} \int_\Omega (K - S_T)\mathbf{1}_{\{S_T(\omega) \leq K\}}\, dP(\omega), \end{aligned} \tag{1}$$

where r is the constant riskless interest rate, E is the risk neutral expectation operator, $\mathbf{1}_{\{S_T \leq K\}}$ is the indicator function for the event set $\{S_T \leq K\}$ and $dP(\omega)$ is the probability measure over the domain set Ω for the random variable S_T. Applying the relation

$$(K - S_T)\mathbf{1}_{\{S_T(\omega) \leq K\}} = \int_0^K \mathbf{1}_{\{S_T(\omega) \leq \xi\}}\, d\xi, \tag{2}$$

we obtain

$$\begin{aligned} \text{put value} &= e^{-r\tau} \int_\Omega \int_0^K \mathbf{1}_{\{S_T(\omega) \leq \xi\}}\, d\xi dP(\omega) \\ &= e^{-r\tau} \int_0^K \int_\Omega \mathbf{1}_{\{S_T(\omega) \leq \xi\}}\, dP(\omega) d\xi \quad \text{(by Fubini's theorem)} \\ &= e^{-r\tau} \int_0^\infty P_r(S_T \leq \xi < K)\, d\xi \\ &= e^{-r\tau} \int_0^K P_r(S_T \leq \xi)\, d\xi. \end{aligned} \tag{3}$$

It may be instructive to provide the following financial interpretation for the above formula. First, we divide the interval $[0, K]$ into n subintervals, each of equal width $\Delta\xi$ so that $n\Delta\xi = K$. The put can be decomposed into the sum of n portfolios, the j^{th} portfolio consists of long holding a put with strike $j\Delta\xi$ and short selling a put with strike $(j-1)\Delta\xi, j = 1, 2, \cdots, n$, where all puts have the same maturity date T. To the leading order in $\Delta\xi$, the value of the j^{th} portfolio is $\{(j\Delta\xi - S_T) - [(j-1)\Delta\xi - S_T]\}P_r(S_T \leq \xi_j), \xi_j = j\Delta\xi$. Taking the limit $n \to \infty$ and $\Delta\xi \to 0$, we obtain

$$\text{put value} = e^{-r\tau} \sum_{j=1}^\infty P_r(S_T \leq \xi_j)\Delta\xi = e^{-r\tau} \int_0^K P_r(S_T \leq \xi)\, d\xi. \tag{4}$$

These n portfolios can be visualized as appropriate replenishments to the sub-replicating portfolio so that the writer of the call option is immunized from possible loss at the maturity of the option. To refine the argument, we examine the role of each of the n portfolios. With the addition of the n^{th} portfolio [long a put with strike K and short a put with strike $(K - \Delta\xi)$] into the sub-replicating portfolio, the writer faces a loss only when S_T falls below $K - \Delta\xi$. Deductively, the protection over the interval $[(j-1)\Delta\xi, j\Delta\xi]$ in the out-of-the-money region of the call is gained with the addition of the j^{th} portfolio. One then proceeds one by one from the n^{th} portfolio down to the 1^{st} portfolio so that the protection over $[0, K]$ is achieved. With the acquisition of all these replenishing portfolios, the writer is immunized from any possible loss at option's maturity even the call expires out-of-the-money. The cost of acquiring all these n portfolios is called the replenishing premium, and its value is given by the integral in equation (3).

A new viable approach in derivative pricing then emerges. The value of an option is given by the sum of the value of the sub-replicating portfolio and the replenishing premium. The pricing of an option amounts to an ingenious choice of the sub-replicating portfolio and the construction of the appropriate replenishing strategy.

The choice of the sub-replicating portfolio is not unique. Suppose the writer of the call option chooses the sub-replicating portfolio to be the null (empty) portfolio, then the replenishment is obtained by taking the collection of infinitely many portfolios, where the j^{th} portfolio consists of long holding of a call with strike $K + j\Delta\xi$ and short selling of a call with strike $K + (j+1)\Delta\xi, j = 1, 2, \cdots$. The replenishing premium is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \sum_{j=1}^{\infty} P_r(S_T > K + j\Delta\xi)\Delta\xi \\ &= e^{-r\tau} \int_0^{\infty} P_r(S_T > \xi > K)\, d\xi \\ &= e^{-r\tau} \int_K^{\infty} P_r(S_T > \xi)\, d\xi. \end{aligned} \tag{5}$$

Since the sub-replicating portfolio has been chosen to be the null portfolio, the call value is then equal to the replenishing premium as defined in equation (5).

In the above formulations, it is not necessary to restrict the random asset price process to the usual lognormal process. Provided that $P_r(S_T \le \xi)$ or $P_r(S_T > \xi)$ for the specified asset price process is given, the integral in equation (4) or equation (5) can be evaluated accordingly.

2.2 One-asset fixed strike and floating strike lookbacks

In this subsection, we would like to demonstrate the robustness of the sub-replication and replenishment approach by pricing the European style one-asset floating-strike and fixed-strike lookback options under continuous and discrete monitoring of the extremum value of the asset price process. Our derivation procedure will be seen to be more direct, intuitive and simple compared to earlier methods reported in the literature (Conze and Viswanathan, 1991). The experience gained in the one-asset pricing models will be beneficial to the development of efficient pricing procedures for the multi-state lookback options.

Continuously monitored floating strike lookback call options

We let $[T_0, T]$ be the continuously monitored period for the minimum value of the asset price process. It is assumed that the current time t is within the monitoring period so that $T_0 < t < T$, and that the period of monitoring ends with the maturity of the lookback call option. Let S_u denote the asset price at time $u, T_0 \leq u \leq T$. Let $\underline{S}[t_1, t_2]$ denote the realized minimum value of the asset price over the period $[t_1, t_2]$. The terminal payoff of the continuously monitored floating strike lookback call option is given by

$$c_{f\ell}(S_T, T) = S_T - \underline{S}[T_0, T]. \tag{6}$$

Note that the realized minimum value of S_u from the earlier time T_0 to the current time t (denoted by $\underline{S}[T_0, t]$) is already known. It is seen that

$$\underline{S}[T_0, T] = \min(\underline{S}[T_0, t], \underline{S}[t, T]). \tag{7}$$

Here, $\underline{S}[t, T]$ is a stochastic state variable with dependence on $S_u, u \in [t, T]$.

First, it seems natural to choose the sub-replicating instrument to be a forward with the same maturity and delivery price $\underline{S}[T_0, t]$. The terminal payoff of the sub-replicating instrument is below that of the forward only when $\underline{S}[t, T] < \underline{S}[T_0, t]$; otherwise, the terminal payoffs of the forward and lookback call are equal. Here, $\underline{S}[t, T]$ is the random variable that determines the occurrence of under replication. Following similar argument as in equation (4), except that S_T is replaced by $\underline{S}[t, T]$, the required replenishing premium to compensate for the occurrence of under replication is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \sum_{j=1}^{\infty} P_r(\underline{S}[t, T] \leq \xi_j) \Delta\xi \\ &= e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P_r(\underline{S}[t, T] \leq \xi) \, d\xi. \end{aligned} \tag{8}$$

The replenishing strategy is to purchase a series of portfolios so as to gain protection in the interval where $\underline{S}[t, T] \leq \underline{S}[T_0, t]$. The value of the continuously monitored European floating strike lookback call option at the current time is given by the sum of the sub-replicating portfolio and the replenishing premium. This gives

$$\begin{aligned} c_{f\ell}(S, t; \underline{S}[T_0, t]) &= S - e^{-r\tau} \underline{S}[T_0, t] + e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P_r(\underline{S}[t, T] \leq \xi) \, d\xi \\ &= S - e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P_r(\underline{S}[t, T] > \xi) \, d\xi, \end{aligned} \tag{9}$$

where S is the current asset price and $S - e^{-r\tau} \underline{S}[T_0, t]$ is the current value of the forward with delivery price $\underline{S}[T_0, t]$ and maturity date T. The probability $P_r[\underline{S}[t, T] > \xi]$ is related to the distribution function for the restricted asset price process with the down barrier ξ.

The choice of the sub-replicating portfolio is not unique. Suppose we choose the sub-replicating instrument to be a European call option with the same maturity and strike price $\underline{S}[T_0, t]$, the terminal payoff of this sub-replicating vanilla call is below that of the floating strike lookback call only when $\underline{S}[t, T] < \min(S_T, \underline{S}[T_0, t])$. To apply the formulation as stated in equation (3), we take

$\underline{S}[t,T]$ as the stochastic state variable that determines under replication or otherwise at maturity. Now, the *effective strike price* is the stochastic quantity $\min(S_T, \underline{S}[T_0,t])$ [note that the same role is taken by K is equation (3)]. The required replenishing premium is then given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau}\int_0^\infty P_r(\underline{S}[t,T] \le \xi < \min(S_T, \underline{S}[T_0,t])\; d\xi \\ &= e^{-r\tau}\int_0^{\underline{S}[T_0,t]} P_r(\underline{S}[t,T] \le \xi < S_T)\; d\xi. \end{aligned} \tag{10}$$

Let $c(S,t;\underline{S}[T_0,t])$ denote the current value of the European vanilla call with strike $\underline{S}[T_0,t]$. It is seen that

$$c_{f\ell}(S,t;\underline{S}[T_0,t]) = c(S,t;\underline{S}[T_0,t]) + e^{-r\tau}\int_0^{\underline{S}[T_0,t]} P_r(\underline{S}[t,T] \le \xi < S_T)\; d\xi. \tag{11}$$

By making use of the relations

$$c(S,t;\underline{S}[T_0,t]) = S - \underline{S}[T_0,t]e^{-r\tau} + e^{-r\tau}\int_0^{\underline{S}[T_0,t]} P_r(S_T \le \xi)\; d\xi \tag{12a}$$

and

$$P_r(\underline{S}(t,T) \le \xi < S_T) = P_r(\underline{S}(t,T) \le \xi) - P_r(S_T \le \xi), \tag{12b}$$

we observe the equivalence of the two price formulas for the floating strike lookback call as given in equations (9) and (11).

Garman (1992) interpreted the replication of the floating strike lookback call by the rollover strategy that involves the sale of a call with a higher strike and the simultaneous purchase of another call with a lower strike. He argued that whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same expiration date but with a strike price equal to the newly established minimum value. Since the call with a lower strike is always more expensive, some extra cost should be charged for the buyer of the lookback call for holding this strike bonus right. It is interesting to see that the *strike bonus premium* happens to be the replenishing premium defined in equation (10).

Continuously monitored fixed strike lookback call option

Let $\overline{S}[t_1,t_2]$ denote the realized maximum value of the asset price over the period $[t_1,t_2]$. The terminal payoff of the continuously monitored floating strike lookback call option is given by

$$c_{fix}(S_T,T) = (\overline{S}[T_0,T] - K)^+ \tag{13}$$

where K is the strike price and x^+ signifies $\begin{cases} x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$. When the fixed strike lookback call expires at-the-money or in-the-money, $\overline{S}[T_0,T] \ge K$, the terminal payoff can be expressed as $(\overline{S}[T_0,T] - S_T) + (S_T - K)$, which is the sum of the terminal payoffs of a floating strike put and a forward on the same asset having delivery price K. It then becomes natural to choose the sub-replicating portfolio to be the sum of the European floating strike lookback put and the forward.

Since $\overline{S}[T_0, T] \geq \overline{S}[T_0, t]$, the fixed strike lookback call is guaranteed to expire in-the-money if it is currently in-the-money. Therefore, when $\overline{S}[T_0, t] \geq K$, the sub-replicating portfolio is guaranteed to be a full replication. On the other hand, when $\overline{S}[T_0, t] < K$ (the fixed strike lookback call is currently out-of-the-money), the sub-replicating portfolio would expire with payoff below that of the fixed strike lookback call when $\overline{S}[T_0, T] < K$. Similar to the reasoning as that used in equation (8), when $\overline{S}[T_0, t] < K$, the replenishing premium to compensate for under replication is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \int_0^K P_r(\max(\overline{S}[T_0, t], \overline{S}[t, T]) \leq \xi)\, d\xi \\ &= e^{-r\tau} \int_{\overline{S}[T_0, t]}^K P_r(\overline{S}[t, T] \leq \xi)\, d\xi. \end{aligned} \tag{14}$$

If we write $p_{f\ell}(S, t; L)$ to denote the current value of the continuously monitored European floating strike lookback put option, where L denotes the current realized maximum asset value, then the above replenishing premium can be expressed as $p_{f\ell}(S, t; K) - p_{f\ell}(S, t; \overline{S}[T_0, t])$.

In summary, the current value of the continuously monitored European fixed strike lookback call option is given by

(i) $\overline{S}[T_0, t] \geq K$

$$c_{fix}(S, t; \overline{S}[T_0, t]) = p_{f\ell}(S, t; \overline{S}[T_0, t]) + S - Ke^{-r\tau}. \tag{15a}$$

(ii) $\overline{S}[T_0, t] < K$

$$c_{fix}(S, t; \overline{S}[T_0, t]) = p_{f\ell}(S, t; K) + S - Ke^{-r\tau}. \tag{15b}$$

Discretely monitored floating strike lookback call options

Suppose the monitoring of the minimum value of the asset price takes place only at discrete instants $t_j, j = 1, 2, \cdots, n$, where t_n is on or before the maturity date of the lookback call option. Suppose the current time is taken to be within $[t_k, t_{k+1})$. The terminal payoff of the discretely monitored floating strike lookback call option is given by

$$c_{f\ell}^{dis}(S_T, T) = S_T - \min(S_{t_1}, S_{t_2}, \cdots, S_{t_n}). \tag{16}$$

We use the notation $\underline{S}[i, j]$ to denote $\min(S_{t_i}, S_{t_{i+1}}, \cdots, S_{t_j}), j > i$. At the current time, $\underline{S}[1, k] = \min(S_{t_1}, S_{t_2}, \cdots, S_{t_k})$ is already known. Similar to the continuously monitored case, we choose the sub-replicating instrument to be a forward with the same maturity date T and delivery price $\underline{S}[1, k]$.

If $\underline{S}[1, k] \leq \underline{S}[k+1, n]$, then the forward expires with the same payoff as that of the discretely monitored European floating strike lookback call; otherwise, the sub-replicating forward expires with payoff below that of the lookback call. Similar to the continuously monitored case, the replenishing premium to compensate for under replication is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \int_0^\infty P_r(\min(\underline{S}[1, k], \underline{S}[k+1, n]) \leq \xi)\, d\xi \\ &= e^{-r\tau} \int_0^{\underline{S}[1, k]} P_r(\underline{S}(k+1, n] \leq \xi)\, d\xi. \end{aligned} \tag{17}$$

The probability function $P_r[\underline{S}[k+1,n] \leq \xi]$ involves the joint distribution of the $n-k$ asset prices at $t_{k+1}, \cdots, t_n$. The value of the discretely monitored European floating strike lookback call option is then given by

$$c_{f\ell}^{dis}(S,t;\underline{S}[1,k]) = S - e^{-r\tau}\underline{S}[1,k] + e^{-r\tau}\int_0^{\underline{S}[1,k]} P_r(\underline{S}[k+1,n] \leq \xi)\ d\xi. \tag{18}$$

3. Lookback spread options

In this section, we would like to apply the technique of sub-replication and replenishment to derive the price formulas of the one-asset and two-asset European lookback spread options. Traders may use the lookback spread options to hedge an existing position that is sensitive to price volatility or to bet on price volatility.

3.1 One-asset lookback spread option

The terminal payoff of an one-asset lookback spread option is given by

$$c_{sp}(S_T,T;K) = (\overline{S}[T_0,T] - \underline{S}[T_0,T] - K)^+, \tag{19}$$

where $\overline{S}(T_0,T)$ and $\underline{S}[T_0,T]$ are the realized maximum value and minimum value of the asset price over $[T_0,T]$, and K is the strike price. From the above terminal payoff structure, a convenient choice of the sub-replicating portfolio would consist of long holding of European lookback call and lookback put, both of floating strike, and short holding of a riskless bond of par value K, all of them have the same maturity as that of the lookback spread option. The terminal payoff of the sub-replicating portfolio is $\overline{S}[T_0,T] - \underline{S}[T_0,T] - K$. It is observed that

$$\begin{aligned} \overline{S}[T_0,T] - \underline{S}[T_0,T] - K &= \max(\overline{S}[T_0,t], \overline{S}[T_0,T]) - \min(\underline{S}[T_0,t], \underline{S}[t,T]) - K \\ &\geq \overline{S}[T_0,t] - \underline{S}[T_0,t] - K, \end{aligned} \tag{20}$$

so the lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money. In this case, the sub-replication is a full replication since the terminal payoffs of the sub-replicating portfolio and the lookback spread option are equal. However, if the lookback spread option is currently out-of-the-money, the terminal payoff of the sub-replicating portfolio would be less than that of the lookback spread option if the lookback spread option expires out-of-the-money, that is,

$$\max(\overline{S}[T_0,t], \overline{S}[t,T]) - \min(\underline{S}[T_0,t], \underline{S}[t,T]) - K < 0. \tag{21}$$

Suppose we treat $\max(\overline{S}[T_0,t], \overline{S}[t,T])$ as the stochastic state variable that determines under replication or otherwise, and $\min(\underline{S}[T_0,t], \underline{S}[t,T]) + K$ as the effective strike price, the required replenishing premium is then given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau}\int_0^\infty P_r(\max(\overline{S}[T_0,t], \overline{S}[t,T]) \leq \xi < \min(\underline{S}[T_0,t], \underline{S}[t,T]) + K)\ d\xi \\ &= e^{-r\tau}\int_{\overline{S}[T_0,t]}^{\underline{S}[T_0,t]+K} P_r(\overline{S}[t,T] \leq \xi < \underline{S}[t,T] + K)\ d\xi. \end{aligned} \tag{22}$$

In summary, the current value of the one-asset European lookback spread option is given by

(i) $\overline{S}[T_0,t] - \underline{S}[T_0,t] - K \geq 0$

$$c_{sp}(S,t;\overline{S}[T_0,t],\underline{S}[T_0,t]) = c_{f\ell}(S,t;\underline{S}[T_0,t]) + p_{f\ell}(S,t;\overline{S}[T_0,t]) - Ke^{-r\tau} \tag{23a}$$

(ii) $\overline{S}[T_0,t] - \underline{S}[T_0,t] - K < 0$

$$\begin{aligned} c_{sp}(S,t;\overline{S}[T_0,t],\underline{S}[T_0,t]) &= c_{f\ell}(S,t;\underline{S}[T_0,t]) + p_{f\ell}(S,t;\overline{S}[T_0,t) - Ke^{-r\tau} \\ &\quad + e^{-r\tau}\int_{\overline{S}[T_0,t]}^{\underline{S}[T_0,t]+K} P_r(\overline{S}[t,T] \leq \xi < \underline{S}[t,T]+K)\ d\xi \end{aligned} \tag{23b}$$

3.2 Two-asset lookback spread option

Let $S_{1,u}$ and $S_{2,u}$ denote the price process of asset 1 and asset 2, respectively. Similarly, we write $\overline{S}_1[t_1,t_2]$ and $\underline{S}_2[t_1,t_2]$ as the realized maximum value of $S_{1,u}$ and realized minimum value of $S_{2,u}$ over the period $[t_1,t_2]$, respectively. The terminal payoff of a two-asset lookback spread option is given by

$$c_{sp}(S_{1,T},S_{2,T},T;K) = (\overline{S}_1[T_0,T] - \underline{S}_2[T_0,T] - K)^+, \tag{24}$$

where K is the strike price. Since we can express $\overline{S}_1[T_0,T] - \underline{S}_2[T_0,T] - K$ as $(\overline{S}_1[T_0,T] - S_{1,T}) + (S_{2,T} - \underline{S}_2[T_0,T)] + S_{1,T} - S_{2,T} - K$, so a natural choice of the sub-replicating portfolio would consist of long holding of one European floating strike lookback put on asset 1, one European floating strike lookback call on asset 2, both of floating strike, 1 unit of asset one and short holding of one unit of asset 2 and a riskless bond of par value K. All instruments in the portfolio have the same maturity as that of the two-asset lookback spread option.

Similar to the one-asset counterpart, the two-asset lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money; and under this scenario, the sub-replicating portfolio will expire with a terminal payoff below that of the lookback spread option if the lookback spread option expires out-of-the-money. Following the same argument as used in the one-asset counterpart, the current value of the two-asset European lookback spread option is given by

(i) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K \geq 0$

$$\begin{aligned} c_{sp}(S_1,S_2,t;\overline{S}_1[t_0,T],\underline{S}_2[T_0,t]) &= p_{f\ell}(S_1,t;\overline{S}_1[T_0,t]) + c_{f\ell}(S_2,t;\underline{S}_2[T_0,t]) \\ &\quad + S_1 - S_2 - Ke^{-r\tau} \end{aligned} \tag{25a}$$

(ii) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K < 0$

$$\begin{aligned} c_{sp}(S_1,S_2,t;\overline{S}_1[T_0,t],\underline{S}_2[T_0,t]) &= p_{f\ell}(S_1,t;\overline{S}_1[T_0,t]) + c_{f\ell}(S_2,t;S_2[T_0,t)] \\ &\quad + S_1 - S_2 - Ke^{-r\tau} \\ &\quad + e^{-r\tau}\int_{\overline{S}_1[T_0,t]}^{\underline{S}_2[T_0,t]+K} P_r(\overline{S}_1[t,T] \leq \xi < \underline{S}_2[t,T]+K)\ d\xi. \end{aligned} \tag{25b}$$

Here, S_1 and S_2 are the current value of asset 1 and asset 2, respectively.

4. Semi-lookback options

The terminal payoff of the semi-lookback options depends on the extreme value of the price at one asset and the terminal values of the prices of other assets. We further illustrate the robustness of the sub-replication and replenishment approach by deriving the price formulas of two-asset and multi-asset semi-lookback options.

4.1 Two-asset semi-lookback option

Let $V_{semi}^2(S_1, S_2, t; \overline{S}_2[T_0, t])$ denote the value of the two-asset semi-lookback option whose terminal payoff is given by $\max(\overline{S}_2[T_0, T] - S_{1,T} - K, 0)$. Since we may write $\overline{S}_2[T_0, T] - S_{1,T} - K = (\overline{S}_2[T_0, T] - S_{2,T}) + S_{2,T} - S_{1,T} - K$, the sub-replicating portfolio is chosen to consist of long holding of one European floating strike lookback put and one unit of asset 2, and short holding of one unit of asset 1 and a riskless bond of par value K, all instruments having the same maturity. The sub-replicating portfolio will expire with a terminal payoff below that of the two-asset semi-lookback option if

$$\max(\overline{S}_2[T_0, t]\overline{S}_2[t, T]) - S_{1,T} - K < 0. \tag{26}$$

Following similar argument as used in equation (22), the required replenishing premium is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \int_0^\infty P_r(\max(\overline{S}_2[T_0, t], \overline{S}_2[t, T]) \leq \xi < S_{1,T} + K)\ d\xi \\ &= e^{-r\tau} \int_{\overline{S}_2[T_0, t]}^\infty P_r(\overline{S}_2[t, T] \leq \xi < S_{1,T} + K)\ d\xi. \end{aligned} \tag{27}$$

The value of this two-asset semi-lookback option is then given by

$$\begin{aligned} V_{semi}^2(S_1, S_2, t; \overline{S}_2[T_0, t] &= p_{f\ell}(S_2, t; \overline{S}_2[T_0, t]) + S_2 - S_1 - Ke^{-r\tau} \\ &\quad + \ e^{-r\tau} \int_{\overline{S}_2[T_0, t]}^\infty P_r(\overline{S}_2[t, T] \leq \xi < S_{1,T} + K)\ d\xi. \end{aligned} \tag{28}$$

4.2 Multi-asset semi-lookback option

Let $V_{semi}^n(S_1, S_2, \cdots, S_n, t; \underline{S}_1[T_0, t])$ denote the value of the multi-asset semi-lookback option whose terminal payoff is given by $\max(\max(S_{2,T}, \cdots, S_{n,T}) - \underline{S}_1[T_0, T], 0)$. From the terminal payoff structure, the value of the sub-replicating portfolio is given by $c_{max}^{n-1}(S_2, \cdots, S_n, t) + c_{f\ell}(S_1, t; \underline{S}_1[T_0, t]) - S_1$, where $c_{max}^{n-1}(S_2, \cdots, S_n, t)$ denotes the value of the $(n-1)$-asset maximum call option with zero strike. Under replication at maturity by the sub-replicating portfolio occurs when

$$\max(S_{2,T}, \cdots, S_{n,T}) < \underline{S}_1[T_0, T] = \min(\underline{S}_1[T_0, t], S_1[t, T]). \tag{29}$$

Following analogous procedure as above,t he value of this n-asset semi-lookback option is given by

$$\begin{aligned} V_{semi}^n(S_1, S_2, \cdots, S_n, t; \underline{S}_1[T_0, t] &= c_{max}^{n-1}(S_2, \cdots, S_n, t) + c_{f\ell}(S_1, t; \underline{S}_1[T_0, t]) - S_1 \\ &\quad + e^{-r\tau} \int_0^{\underline{S}_1[T_0, t]} P_r(\max(S_2, \cdots, s_n) \leq \xi < \underline{S}_1[t, T])\ d\xi. \end{aligned} \tag{30}$$

5. Conclusion

The novel approach of choosing a sub-replicating portfolio and subsequently devising the replenishing premium for pricing exotic options has been developed in this paper. The robustness of the approach has been illustrated through the pricing of several multi-state lookback options. We observe that the use of an elegant financial intuition has significantly reduced the complexity in the analytic pricing procedures of multi-state lookback options.

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Appendix

With the assumption of the lognormal distribution for the asset price processes, we list the probability distributions that occur in the lookback option price formulas derived in the paper. We let $\sigma_i, i = 1, 2$, denote the volatility of the asset price process S_i The dynamics of S_i is given by

$$(A1) \qquad \frac{dS_i}{S_i} = r dt + \sigma_i dZ_i, \qquad i = 1, 2,$$

where dZ_i is the Wiener process and $dZ_1 dZ_2 = \rho dt$. Here, ρ is the correlation coefficient between dZ_1 and dZ_2. We write $X_i(t) = \ln S_i(t)$ so that

$$X_i(t) = \alpha_i t + \sigma_i Z_i(t), \quad i = 1, 2, \tag{A2}$$

is a Brownian motion with drift rate α_i, $\alpha_i = r - \dfrac{\sigma_i^2}{2}$. Further, we define

$$\underline{X}_i(t) = \min_{0 \le u \le t} X_i(u) \qquad \text{and} \qquad \overline{X}_i(t) = \max_{0 \le u \le t} X_i(u). \tag{A3}$$

1. Probability distributions involving single asset

For notational simplicity, we drop the subscript for $X_1(t), \overline{X}_1(t), \underline{X}_1(t), \sigma_1$ and α_1.

$$\begin{aligned} P_r(\underline{X}(t) \ge \underline{x}, X(t) \ge x) &= G(x, \underline{x}, t; \alpha) \\ &= N\left(\frac{-x + \alpha t}{\sigma \sqrt{t}}\right) - e^{\frac{2\alpha \underline{x}}{\sigma^2}} N\left(\frac{-\underline{x} + 2\underline{x} + \alpha t}{\sigma \sqrt{t}}\right) \end{aligned} \tag{A4}$$

$$P_r(\underline{X}(t) \ge x) = G(x, x, t; \alpha) \tag{A5}$$

$$P_r(\overline{X}(t) \le \overline{x}, X(t) \le x) = G(-x, -\overline{x}, t; -\alpha) \tag{A6}$$

$$P_r(\overline{X}(t) \le x) = G(-x, -x, t; -\alpha) \tag{A7}$$

$$\begin{aligned} P_r(\underline{X} \ge x, \overline{X} \le y) = \sum_{n=-\infty}^{\infty} e^{[2n\alpha(y-x)]/\sigma^2} \Bigg\{ \Bigg[& N\left(\frac{y - \alpha t - 2n(y-x)}{\sigma \sqrt{t}}\right) \\ & - N\left(\frac{x - \alpha t - 2n(y-x)}{\sigma_1 \sqrt{t}}\right)\Bigg] \\ & - e^{2\alpha x/\sigma^2} \Bigg[N\left(\frac{y - \alpha t - 2n(y-x) - 2x}{\sigma \sqrt{t}}\right) \\ & - N\left(\frac{x - \alpha t - 2n(y-x) - 2x}{\sigma \sqrt{t}}\right)\Bigg]\Bigg\} \end{aligned} \tag{A8}$$

2. Probability distributions involving two assets

(a) Semi-lookback options

$$\begin{aligned} & P_r(\underline{X}_1(t) \ge \underline{x}_1, X_1(t) \ge x_1, X_2(t) \le x_2) \\ &= G_{semi}(x_1, x_2, \underline{x}_1, t; \alpha_1, \alpha_2, \rho) \\ &= N_2\left(\frac{-x_1 + \alpha_1 t}{\sigma_1 \sqrt{t}}, \frac{x_2 - \alpha_2 t}{\sigma_2 \sqrt{t}}; -\rho\right) - e^{\frac{2\alpha_1 \underline{x}_1}{\sigma_1^2}} N_2\left(\frac{-x_1 + \underline{x}_1 + \alpha_1 t}{\sigma_1 \sqrt{t}}, \frac{x_2 - \alpha_2 t}{\sigma_2 \sqrt{t}}; -\rho\right) \end{aligned} \tag{A9}$$

$$P_r(\underline{X}_1(t) \ge \underline{x}_1, X_2(t) \le x_2) = G_{semi}(\underline{x}_1, x_2, \underline{x}_1, t; \alpha_1, \alpha_2, \rho) \tag{A10}$$

$$P_r(\overline{X}_1(t) \le \overline{x}_1, X_1(t) \le x_1, X_2(t) \le x_2) = G_{semi}(-x_1, x_2, -\overline{x}_1, t; -\alpha_1, \alpha_2, -\rho) \tag{A11}$$

$$P_r(\overline{X}_1(t) \le \overline{x}_1, X_2(t) \le x_2) = G_{semi}(-\overline{x}_1, x_2, -\overline{x}_1, t; -\alpha_1, \alpha_2, -\rho) \tag{A12}$$

(b) Double-lookback options

$$\begin{aligned} P_r(\overline{X}_1(t) \le x_1, \overline{X}_2(t) \le x_2) &= G_2(x_1, x_2, t; \alpha_1, \alpha_2, \rho) \\ &= e^{a_1 x_1 + a_2 x_2 + bt} f(r', \theta' t) \end{aligned} \tag{A13}$$

where $$a_1 = \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \qquad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}$$

$$b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2$$

$$f(r',\theta',t) = \frac{2}{\alpha' t}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta'}{\alpha'}\right) e^{-r'^2/2t} \int_0^{\alpha'} \sin\left(\frac{n\pi\theta}{\alpha'}\right) g_n(\theta)\ d\theta$$

$$g_n(\theta) = \int_0^{\infty} r e^{-r^2/2t} e^{-b_1 r\cos(\theta-\alpha) - b_2 r \sin(\theta-\alpha)} I_{(n\pi)/\alpha}\left(\frac{rr'}{t}\right)\ dr$$

$$r' = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)^{1/2}, \quad \theta' = \theta + \alpha$$

$$\cos\theta = \frac{x_1}{\sigma_1 r}, \quad \tan\alpha = \frac{\rho}{\sqrt{1-\rho^2}}, \quad \alpha' = \alpha + \frac{\pi}{2},$$

$$b_1 = a_1\sigma_1 + a_2\sigma_2\rho, \quad b_2 = a_2\sigma_2\sqrt{1-\rho^2}.$$

$(A14)$ $$P_r(\overline{X}_1 \le x_1, \underline{X}_2(t) \ge x_2) = G_2(x_1, -x_2, t; \alpha_1, -\alpha_2, -\rho)$$

$(A15)$ $$P_r(\underline{X}_1 \ge x_1, \underline{X}_2(t) \le x_2) = G_2(-x_1, -x_2, t; -\alpha_1, -\alpha_2, -\rho)$$