Multi-asset barrier options and occupation time derivatives

by

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Date of completion: 28 February, 2001

Abstract

We formulate a general framework to price various forms of multi-asset barrier options and occupation time derivatives with one state variable having the barrier feature. Based on the lognormal assumption of the asset price processes, we develop the splitting direction technique for deriving the joint density functions of multi-variate terminal asset prices with provision of barriers. Compared to the reflection principle in statistical approach and the method of images in partial differential equation approach, our formulation greatly facilitates the derivation of the price formulas of multi-asset barrier options and occupation time derivatives. By following a unified procedure, we illustrate that the multi-asset option price formulas can be deduced in a systematic manner as extensions from those of their one-asset counterparts. Our formulation has been successfully applied to derive the analytic price formulas of multi-asset options with external two-sided barriers and sequential barriers, multi-asset step options and delayed barrier options.

1. INTRODUCTION

Barrier options have become so popular in the financial markets that they are no longer considered as exotic options. The inclusion of a barrier provision in the option contract allows the investor to eliminate those unlikely scenarios as viewed by herself, thus achieving option premium reduction. The analytical valuation of the down-and-out call option first appeared in the seminal paper by Merton (1973). Since then there are numerous articles which consider the pricing of different forms of barrier options (Rubinstein and Reiner, 1991; Rich, 1994). The barrier provision may take more exotic forms, such as two-sided barriers (Kunitomo and Ikeda, 1992), sequential barriers (Sidenius, 1998) and external barrier (Heynen and Kat, 1994).

The barrier feature is well known to have the undesirable "circuit breaker" effect. When evaluated at the barrier, the barrier option's delta is discontinuous and option's gamma tends to

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infinite value, thus causing serious hedging difficulties for option writers (Linetsky, 1999).

Gradual knock-out options are introduced to modify the abrupt one-touch knock-out feature in traditional barrier options. These options are parameterized by the occupation time, which is the total time spent by the asset price process staying in the knock-out region. Hence, these gradual knock-out options are also called occupation time derivatives. The improvement from the risk management perspective of these occupation time derivatives over the one-touch barrier option is well explored in Linetsky's paper (1999). Using the technique of Laplace transform, Linetsky obtained the analytic formulas for these one-asset gradual knock-out options. The price formulas of other one-asset occupation time derivatives were also obtained by Douady (1998) and Hugonnier (1999).

Option models which are multi-variate in nature are abundant in the financial markets. For multi-state options, the option value is determined by the stochastic behaviors of several underlying asset price and / or stochastic variables (like interest rates) and the correlation coefficients between these stochastic quantities. Under the Black-Scholes assumption of lognormality of the asset price processes, the option value is governed by a multi-dimensional parabolic diffusion type equation. Unlike the usual diffusion type equations, the multi-dimensional Black-Scholes option equation contains second order cross derivative terms due to the non-vanishing of the correlation coefficients among the stochastic state variables.

The analytical valuation of the option price function amounts to the determination of the transition density function of the terminal asset prices conditional on the values of the current asset prices. For most one-asset barrier option models, the transition density functions can be found quite easily using the reflection principle or the method of images. For multi-state models, the transition density of the terminal asset prices for the unrestricted processes can be obtained without great difficulty. However, the integration of the expectation integrals can be quite tedious. By following an ingenious method of choosing a set of appropriate similarity variables, Johnson (1987) managed to obtain the price formulas for various European multi-asset vanilla option models. The price formulas of multi-asset options with one-sided external barrier have also been obtained by Heynen and Kat (1994), Rich and Leipus (1997) and Kwok *et al.* (1998).

It is almost analytically intractable to price the multi-state occupation time derivatives by extending the techniques used by Linetsky (1999), Douady (1998) and Hugonnier (1999) in their pricing frameworks for one-asset option models. In this paper, we formulate the splitting direction technique to derive the transition density functions of the restricted asset price processes associated with the presence of barriers. With this robust formalism, one can deduce in a straightforward manner the price formulas for the multi-asset version of occupation time derivatives from those of their one-asset counterparts.

The paper is organized as follows. In the next section, we state the theorems which formulate the splitting direction technique. The various forms of splitting are achieved by taking suitable forms on the transformation of the dependent and independent variables. In Section 3, we apply the splitting direction technique to derive the price formulas of multi-asset options with two-sided barriers and sequential barriers. These price formulas are represented in the most succinct forms. In Section 4, we develop the general pricing methodology for occupation time derivatives with separable barrier variable and payoff variables. The paper ends with conclusive remarks in the last section.

2. FORMULATION OF THE SPLITTING DIRECTION TECHNIQUE

It is well known that the presence of the drift terms in the governing equation of a multi-asset option model is the primary source of complication in the derivation procedure of finding the fundamental solution of the differential equation. In this section, we present several theorems which show how to decompose the governing equation into simpler structures. Theorem 1 states the result on the use of a transformation on the dependent variable such that the governing equation is reduced to one without the drift terms. Next, the splitting direction technique is summarized in Theorem 2. By adopting an appropriate transformation of the independent variables, we can split the dependent variable as a product of two dependent variables, one is dependent on single independent variable and the other is dependent on the other remaining independent variables. An important mathematical identity that is useful in the derivation procedures in later sections is obtained as a corollary of Theorem 2. Lastly, a generalized form of the Girsanov Theorem is presented in Proposition 3.

Theorem 1 (Separating the drift terms)

If ϕ satisfies the following forward Fokker-Planck equation governing the density function of multi-variate unrestricted joint normal processes with drifts

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \phi}{\partial x_i \partial x_j} - \sum_{j=1}^{n}\mu_j\frac{\partial \phi}{\partial x_j}, \quad t>0, -\infty<x_j<\infty, j=1,2,\cdots,n, \tag{2.1}$$

then ϕ can be decomposed into the product of Q and ψ

$$\phi = Q\psi \tag{2.2}$$

where

$$\begin{aligned} Q &= \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\boldsymbol{\mu}t) - \mathbf{x}^T R^{-1}\mathbf{x}}{2t}\right) \\ &= e^{\boldsymbol{\mu}R^{-1}\boldsymbol{\xi}}\exp\left(-\frac{(\mathbf{x}-\boldsymbol{\xi}-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\boldsymbol{\xi}-\boldsymbol{\mu}t) - (\mathbf{x}-\boldsymbol{\xi})^T R^{-1}(\mathbf{x}-\boldsymbol{\xi})}{2t}\right), \end{aligned} \tag{2.3}$$

and ψ satisfies the following simplified diffusion equation without the drift terms

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j}, \quad t>0, -\infty<x_j<\infty, \quad j=1,2,\cdots,n. \tag{2.4}$$

Here, $\mathbf{x}=(x_1\cdots x_n)^T, \boldsymbol{\mu}=(\mu_1\cdots\mu_n)^T, R$ is the correlation coefficient matrix whose $(i,j)^{\text{th}}$ entry is $\rho_{ij}, i,j=1,2,\cdots,n$. The vector $\boldsymbol{\xi}$ in Eq. (2.3) can be chosen arbitrarily.

The proof of Theorem 1 is argued as follows. The factor Q can be considered as a transformation on the dependent variable ϕ. In order to eliminate the drift terms in Eq. (2.1), the usual procedure is to seek a transformation of the form

$$\phi(\mathbf{x},t) = e^{p_1x_1+\cdots+p_nx_n+qt}\psi(\mathbf{x},t). \tag{2.5}$$

Rather than deriving Q by resorting to the direct substitution into the governing equation, we apply the following argument to deduce an elegant representation for Q. The free space fundamental solution of Eq. (2.4) is $\psi(\mathbf{x}, t) = \frac{1}{(2\pi t)^{n/2}} \frac{1}{\sqrt{\det R}} \exp\left(-\frac{\mathbf{x}^T R^{-1} \mathbf{x}}{2t}\right)$ while that of Eq. (2.1) is $\phi(\mathbf{x}, t) = \frac{1}{(2\pi t)^{n/2}} \frac{1}{\sqrt{\det R}} \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\boldsymbol{\mu}t)}{2t}\right)$. These two fundamental solutions must observe the relation (2.5), and taking their ratio leads to the first form for Q in Eq. (2.3). The second form for Q contains the dummy vector variable $\boldsymbol{\xi}$. This particular form is useful when we apply a shifting transformation on the independent variable $\mathbf{x}$.

Theorem 2 (Splitting direction technique)
If ϕ_n satisfies the forward Fokker-Planck equation with semi-infinite range in the first independent variable

$$\frac{\partial \phi_n}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \phi_n}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \mu_j \frac{\partial \phi_n}{\partial x_j},$$

$$t > 0, b_1 < x_1 < \infty, -\infty < x_j < \infty, j = 2, \cdots, n, \tag{2.6}$$

then the following linear transformation of the independent variables

$$z_j = \begin{cases} x_1 & \text{if } j = 1 \\ \dfrac{x_j - \rho_{1j} x_1}{\sqrt{1-\rho_{1j}^2}} & \text{if } j = 2, 3, \cdots, n \end{cases} \tag{2.7}$$

leads to the splitting of ϕ_n in the following sense

$$\phi_n(z_1, z_2, \cdots, z_n, t) = \phi_1(z_1, t)\phi_{n-1}(z_2, \cdots, z_n, t). \tag{2.8}$$

The reduced functions $\phi_1(z_1, t)$ and $\phi_{n-1}(z_2, \cdots, z_n, t)$ satisfy, respectively, the following equations

$$\frac{\partial \phi_1}{\partial t} = \frac{1}{2}\frac{\partial^2 \phi_1}{\partial z_1^2} - \mu_1 \frac{\partial \phi_1}{\partial z_1}, \quad t > 0, b_1 < z_1 < \infty, \tag{2.9}$$

$$\frac{\partial \phi_{n-1}}{\partial t} = \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n} \widetilde{\rho}_{ij} \frac{\partial^2 \phi_{n-1}}{\partial z_i \partial z_j} - \sum_{j=2}^{n} \widetilde{\mu}_j \frac{\partial \phi_{n-1}}{\partial z_j}, \quad t > 0, -\infty < z_j < \infty, j = 2, \cdots, n, \tag{2.10a}$$

where

$$\widetilde{\rho}_{ij} = \frac{\rho_{ij} - \rho_{1i}\rho_{1j}}{\sqrt{(1-\rho_{1i}^2)(1-\rho_{1j}^2)}} \quad \text{and} \quad \widetilde{\mu}_j = \frac{\mu_j - \rho_{1j}\mu_1}{\sqrt{1-\rho_{1j}^2}}, \quad i, j = 2, 3, \cdots, n. \tag{2.10b}$$

This splitting direction technique is particularly useful to deal with multi-state option models where only one state variable (say x_1) has the barrier feature. Now, the barrier variable z_1 ($z_1 = x_1$ as defined) is uncorrelated with $z_2, \cdots, z_n$, by virtue of the transformation given in Eq. (2.7). The proof of Theorem 2 is given in Appendix A.

Corollary

Let $\psi_n(\mathbf{x}, t; R)$ denote the fundamental solution of Eq. (2.4), that is

$$\psi_n(\mathbf{x}, t; R) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det R}} \exp\left(-\frac{\mathbf{x}^T R^{-1}\mathbf{x}}{2t}\right). \tag{2.11}$$

Write $\widetilde{\mathbf{z}} = (z_2 \cdots z_n)^T$, where $z_2, \cdots, z_n$ are related to $x_1, \cdots, x_n$ by Eq. (2.7); and $\widetilde{R}$ is the $(n-1) \times (n-1)$ correlation coefficient matrix whose entries are $\widetilde{\rho}_{ij}, i, j = 2, \cdots, n$. We then have

$$\psi_n(\mathbf{x} - \alpha R\mathbf{e}_1, t; R) = \psi_1(z_1 - \alpha, t)\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R}) \det J, \tag{2.12}$$

where α is any scalar, $\boldsymbol{e}_1 = (1, 0 \cdots 0)^T$ and $J = \dfrac{\partial(z_1, z_2, \cdots, z_n)}{\partial(x_1, x_2, \cdots, x_n)}$. For its proof, see Appendix B.

Proposition 3 (A generalized form of the Girsanov Theorem)

Let $\boldsymbol{X}(t) = (\boldsymbol{X}_1(t) \cdots \boldsymbol{X}_n(t))$ be a vector of Brownian motions under the probability measure $\mathbf{P}_n$ with unit variance rate, that is,

$$d\mathbf{P}_n = \psi_n(\boldsymbol{X}, t; R)\ d\boldsymbol{X}. \tag{2.13}$$

Here, $\psi_n(\boldsymbol{X}, t; R)$ is the fundamental solution as given in Eq. (2.11) and R is the correlation coefficient matrix. For any constant λ, the new vector defined by $\boldsymbol{X} + \lambda t R\boldsymbol{e}_k$ is then a vector of Brownian motions under a new probability measure $\mathbf{Q}_n^k$ that satisfies the following Radon-Nikodym derivative

$$\frac{d\mathbf{Q}_n^k}{d\mathbf{P}_n} = \exp\left(-\lambda X_k(t) - \frac{\lambda^2 t}{2}\right). \tag{2.14}$$

The proof of Proposition 3 is given in Appendix C.

3. MULTI-ASSET OPTIONS WITH EXTERNAL TWO-SIDED BARRIERS

We consider the class of multi-asset option models with an external barrier variable. The barrier variable does not determine the payoff of the option. Rather, it determines whether the option is knocked out when the value of the barrier variable breaches some pre-determined level (one-sided barrier) or stays outside a certain range of values (two-sided barriers).

The valuation of multi-asset options with an external one-sided barrier has been considered by Heynen and Kat (1994), Rich and Leipus (1997) and Kwok *et al.* (1998). In this section, we illustrate how to apply the formalism in Section 2 to derive the price formulas of multi-asset options whose external barrier variable has two-sided barriers.

Let S_1^t denote the value of the barrier variable and S_i^t denote the value of asset $i, i = 2, \cdots, n$, at time t. For the multi-asset maximum call option with an external barrier, the terminal payoff is given by $\max(\max(S_2^T, \cdots, S_n^T) - X, 0)$, where X is the strike price. We adopt the usual Black-Scholes assumptions on the capital market. In the risk neutral world, we assume $S_i^t, i = 1, 2, \cdots, n$ to follow the lognormal diffusion processes

$$\frac{dS_i^t}{S_i^t} = r\ dt + \sigma_i\ dz_i, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

where r is the riskless interest rate, σ_i is the volatility of asset i, dz_i is the Wiener process for asset $i, i = 1, 2, \cdots, n$. Let ρ_{ij} denote the correlation coefficient between dz_i and dz_j. We define

$$x_i = \frac{1}{\sigma_i} \ln \frac{S_i^t}{S_i} \quad \text{and} \quad \mu_i = \frac{r - \frac{\sigma_i^2}{2}}{\sigma_i}, \quad i = 1, 2, \cdots, n, \tag{3.2}$$

where $S_i, i = 1, 2, \cdots, n$ are the asset values at the current time (taken to be the zeroth time). Let H and L denote the upstream and downstream barriers of the barrier variable. The call option will be knocked out when $S_1^t > H$ or $S_1^t < L$ at any time t during the life of the option. We define $M_1 = \frac{1}{\sigma_1} \ln \frac{H}{S_1}$ and $m_1 = \frac{1}{\sigma_1} \ln \frac{L}{S_1}$.

Joint density function with provision of two-sided barrier levels

Let $\Phi(x_1, x_2, \cdots, x_n, t; R)$ denote the density function of the joint process of the asset prices and barrier variable with the provision of two-sided barrier levels on $x_1, m_1 < x_1 < M_1$. To find $\Phi(x_1, x_2, \cdots, x_n, t)$, we apply the splitting direction technique presented in Section 2. First, we consider the following one-dimensional diffusion equation

$$\frac{\partial \phi_1}{\partial t} = \frac{1}{2} \frac{\partial^2 \phi_1}{\partial x_1^2}, \qquad m_1 < x_1 < M_1, t > 0. \tag{3.3}$$

Its fundamental solution is known to be (Kevorkian, 1990)

$$\phi_1(x_1, t) = \sum_{k=-\infty}^{\infty} [\psi_1(x_1 - 2k(M_1 - m_1), t) - \psi_1(x_1 - 2k(M_1 - m_1) - 2m_1, t)], \tag{3.4}$$

where $\psi_1(x_1, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x_1^2}{2t}\right)$. By applying Theorem 2 and its Corollary, the fundamental solution to the following n-dimensional diffusion equation

$$\frac{\partial \phi_n}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \phi_n}{\partial x_i \partial x_j}, \quad m_1 < x_1 < M_1, -\infty < x_j < \infty, j = 2, \cdots, n, t > 0, \tag{3.5}$$

is given by

$$\begin{aligned} \phi_n(\mathbf{x}, t) &= \sum_{k=-\infty}^{\infty} [\psi_1(z_1 - 2k(M_1 - m_1), t) - \psi_1(z_1 - 2k(M_1 - m_1) - 2m_1, t)] \\ &\qquad \psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R}) \det J \\ &= \sum_{k=-\infty}^{\infty} [\psi_n(\mathbf{x} - 2k(M_1 - m_1) R\boldsymbol{e}_1, t; R) \\ &\qquad - \psi_n(\mathbf{x} - 2[k(M_1 - m_1) - m_1] R\boldsymbol{e}_1, t; R)], \end{aligned} \tag{3.6}$$

where $\psi_n(\mathbf{x}, t; R)$ is defined in Eq. (2.11). Let $\Phi(\mathbf{x}, t; R)$ denote the fundamental solution to the following n-dimensional Fokker-Planck equation

$$\begin{aligned} &\frac{\partial \Phi}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \mu_j \frac{\partial \Phi}{\partial x_j}, \\ &\qquad m_1 < x_1 < M_1, \quad -\infty < x_j < \infty, j = 2, \cdots, n, t > 0. \end{aligned} \tag{3.7}$$

By applying Theorem 1 and choosing $\boldsymbol{\xi}$ to be $2k(M_1 - m_1)R\boldsymbol{e}_1$ and $2[k(M_1 - m_1) - m_1]R\boldsymbol{e}_1$ successively in the second form of Q in Eq. (2.3), we obtain

$$\begin{aligned}\Phi(\mathbf{x}, t; R) = \sum_{k=-\infty}^{\infty} \{ & e^{2\mu_1 k(M_1-m_1)}\psi_n(\mathbf{x} - 2k(M_1 - m_1)R\boldsymbol{e}_1 - \boldsymbol{\mu}t, t; R) \\ & - e^{2\mu_1[k(M_1-m_1)-m_1]}\psi_n(\mathbf{x} - 2[k(M_1 - m_1) - m_1]R\boldsymbol{e}_1 - \boldsymbol{\mu}t, t; R)\}. \end{aligned} \tag{3.8}$$

Option value of maximum call with two-sided external barriers

Following the discounted expectation approach, the value of the multi-asset maximum call option with two-sided external barrier levels H and L is given by

$$\begin{aligned} V(S_1, S_2, \cdots, S_n, T) = e^{-rT} \int_{m_1}^{M_1} \int_{D_{n-1}} & \Phi(x_1, x_2, \cdots, x_n, T; R) \\ & \max(\max(S_2e^{\sigma_2 x_2}, \cdots, S_n e^{\sigma_n x_n}) - X, 0)\, dx_n \cdots dx_2 dx_1, \end{aligned} \tag{3.9}$$

where T is the expiry time and D_{n-1} is the domain in the $(n-1)$-dimensional $(x_2, \cdots, x_n)$-plane inside which $\max(S_2e^{\sigma_2 x_2}, \cdots, S_n e^{\sigma_n x_n}) > X$ is satisfied. Correspondingly, we let D_{n-1}^{ℓ} denote the domain inside which $S_\ell e^{\sigma_\ell x_\ell}$ being the maximum among the $n-1$ quantities $S_2e^{\sigma_2 x_2}, \cdots, S_n e^{\sigma_n x_n}$. The representation of D_{n-1}^{ℓ} is given by

$$\begin{aligned} D_{n-1}^{\ell} = \bigg\{ & (x_2, \cdots, x_n) : x_\ell \geq \frac{1}{\sigma_\ell} \ln \frac{X}{S_\ell}, \\ & x_\ell \geq \frac{\sigma_j}{\sigma_\ell} x_j - \frac{1}{\sigma_\ell} \ln \frac{S_\ell}{S_j}, j = 2, \cdots, n \text{ but } j \neq \ell \bigg\}, \ell = 2, \cdots, n. \end{aligned}$$

The terminal payoff becomes $S_\ell e^{\sigma_\ell x_\ell} - X$ inside the domain D_{n-1}^{ℓ}. The integral in Eq. (3.9) can be decomposed into the sum of $n-1$ integrals. The integration domain of a typical term I_ℓ is $[m_1, M_1] \times D_{n-1}^{\ell}$. Now, I_ℓ is formally represented by

$$\begin{aligned} I_\ell = e^{-rT} \sum_{k=-\infty}^{\infty} \int_{m_1}^{M_1} \int_{D_{n-1}^{\ell}} & (S_\ell e^{\sigma_\ell x_\ell} - X) \\ & [e^{2\mu_1 \alpha_k}\psi_n(\mathbf{x} - 2\alpha_k R\boldsymbol{e}_1 - \boldsymbol{\mu}T, T; R) - e^{2\mu_1 \alpha'_k}\psi_n(\mathbf{x} - 2\alpha'_k R\boldsymbol{e}_1 - \boldsymbol{\mu}T, T; R)] \\ & dx_n \cdots dx_2 dx_1 \end{aligned} \tag{3.10}$$

with $\alpha_k = k(M_1 - m_1)$ and $\alpha'_k = k(M_1 - m_1) - m_1$. To facilitate the integration in I_ℓ, we apply the following linear transformation of the independent variables: $\mathbf{y}^\ell = A^\ell \mathbf{x}$, where

$$y_j^\ell = \begin{cases} x_1 & \text{if } j = 1 \\ -x_\ell & \text{if } j = \ell \\ \dfrac{\sigma_j}{\sigma_{j\ell}} \left(x_j - \dfrac{\sigma_\ell}{\sigma_j} x_\ell \right) & \text{otherwise} \end{cases}, \tag{3.11}$$

with $\sigma_{j\ell}^2 = \sigma_j^2 - 2\rho_{j\ell}\sigma_j\sigma_\ell + \sigma_\ell^2$. The integration domain for I_ℓ becomes

$$(m_1, M_1) \times D_{n-1}^\ell = \left\{ (y_1^\ell, y_2^\ell, \cdots, y_n^\ell) : m_1 < y_1^\ell < M_1, y_\ell^\ell = \frac{1}{\sigma_\ell} \ln \frac{S_\ell}{X}, \right.$$
$$\left. y_j^\ell \le \frac{1}{\sigma_{j\ell}} \ln \frac{S_\ell}{S_j}, \quad j = 2, \cdots, n \text{ and } j \neq \ell \right\}.$$

Also, we have

$$\begin{aligned} &(\mathbf{x} - 2\alpha_k R\boldsymbol{e}_1 - \boldsymbol{\mu}T)^T R^{-1} (\mathbf{x} - 2\alpha_k R\boldsymbol{e}_1 - \boldsymbol{\mu}T) \\ &= (\mathbf{y}^\ell - 2\alpha_k R^\ell \boldsymbol{e}_1 - A^\ell \boldsymbol{\mu}T)^T (R^\ell)^{-1} (\mathbf{y}^\ell - 2\alpha_k R^\ell \boldsymbol{e}_1 - A^\ell \boldsymbol{\mu}T), \end{aligned} \tag{3.12}$$

where $R^\ell = A^\ell R A^{\ell T}$. Consider the typical k^{th} term in I_ℓ [see Eq. (3.10)]

$$\begin{aligned} I_\ell^k &= \int_{m_1}^{M_1} \int_{D_{n-1}^\ell} (S_\ell e^{\sigma_\ell x_\ell} - X) \psi_n(\mathbf{x} - 2\alpha_k R\boldsymbol{e}_1 - \boldsymbol{\mu}T, T; R)\, dx_n \cdots dx_2 dx_1 \\ &= \int_{m_1}^{M_1} \int_{-\infty}^{\frac{1}{\sigma_{2\ell}} \ln \frac{S_\ell}{S_2}} \cdots \int_{-\infty}^{\frac{1}{\sigma_\ell} \ln \frac{S_\ell}{X}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{n\ell}} \ln \frac{S_\ell}{S_n}} (S_\ell e^{-\sigma_\ell y_\ell^\ell} - X) \\ &\qquad \psi_n(\mathbf{y}^\ell - 2\alpha_k R^\ell \boldsymbol{e}_1 - A^\ell \boldsymbol{\mu}T, T; R^\ell)\, dy_n^\ell \cdots dy_2^\ell dy_1^\ell. \end{aligned} \tag{3.13a}$$

By applying the extended form of the Girsanov Theorem (see Proposition 3), we obtain

$$\begin{aligned} I_\ell^k &= S_\ell e^{rT} [e^{2\beta_k^\ell} N(\mathbf{d}_2^\ell - \mathbf{b}_k^\ell; R^\ell) - e^{2\beta_{k'}^\ell} N(\mathbf{d}_4^\ell - \mathbf{b}_{k'}^\ell; R^\ell)] \\ &\quad - X[N(\mathbf{d}_1^\ell - \mathbf{b}_k^\ell; R^\ell) - N(\mathbf{d}_3^\ell - \mathbf{b}_{k'}^\ell; R^\ell)], \end{aligned} \tag{3.13b}$$

where the j^{th} component of $\mathbf{d}_1^\ell$ is

$$d_{1,j}^\ell = \begin{cases} M_1/\sqrt{T} & \text{if } j = 1 \\ \dfrac{1}{\sigma_\ell\sqrt{T}} \ln \dfrac{S_\ell}{X} & \text{if } j = \ell \\ \dfrac{1}{\sigma_{\ell j}\sqrt{T}} \ln \dfrac{S_\ell}{S_j} & \text{otherwise} \end{cases}, \tag{3.14a}$$

$$\mathbf{d}_2^\ell = \mathbf{d}_1^\ell + \sigma_\ell \sqrt{T} R^\ell \boldsymbol{e}_\ell, \quad \mathbf{d}_3^\ell = \mathbf{d}_1^\ell - \frac{M_1 - m_1}{\sigma_1\sqrt{T}} \boldsymbol{e}_1, \quad \mathbf{d}_4^\ell = \mathbf{d}_2^\ell - \frac{M_1 - m_1}{\sigma_1\sqrt{T}} \boldsymbol{e}_1, \tag{3.14b}$$

$$\mathbf{b}_k^\ell = \frac{2\alpha_k R^\ell \boldsymbol{e}_1 + A^\ell \boldsymbol{\mu}T}{\sqrt{T}}, \text{ and } \mathbf{b}_{k'}^\ell = \frac{2\alpha_k' R^\ell \boldsymbol{e}_1 + A^\ell \boldsymbol{\mu}T}{\sqrt{T}}, \ell = 2, \cdots, n, k \text{ is any integer}, \tag{3.14c}$$

$$[r_{ij}^\ell] = R^\ell = A^\ell R A^{\ell^T}, \quad \beta_k^\ell = r_{1\ell}^\ell \sigma_\ell \alpha_k \quad \text{and} \quad \beta_{k'}^\ell = r_{1\ell}^\ell \sigma_\ell \alpha_k'. \tag{3.14d}$$

Finally, the value of the multi-asset maximum call option with two-sided external barrier is found to be

$$\begin{aligned} V(S_1, S_2, \cdots, S_n, T) = \sum_{k=-\infty}^{\infty} \sum_{\ell=2}^{n} S_\ell &\left\{ e^{2\mu_1\alpha_k + 2\beta_k^\ell} \left[N(\mathbf{d}_2^\ell - \mathbf{b}_k^\ell; R^\ell) - N(\mathbf{d}_4^\ell - \mathbf{b}_k^\ell; R^\ell) \right] \right. \\ &\left. - e^{2\mu_1\alpha_k' + 2\beta_{k'}^\ell} \left[N(\mathbf{d}_2^\ell - \mathbf{b}_{k'}^\ell; R^\ell) - N(\mathbf{d}_4^\ell - \mathbf{b}_{k'}^\ell; R^\ell) \right] \right\} \\ - X e^{-rT} &\left\{ e^{2\mu_1\alpha_k + 2\beta_k^\ell} \left[N(\mathbf{d}_1^\ell - \mathbf{b}_k^\ell; R^\ell) - N(\mathbf{d}_3^\ell - \mathbf{b}_k^\ell; R^\ell) \right] \right. \\ &\left. - e^{2\mu_1\alpha_k' + 2\beta_{k'}^\ell} \left[N(\mathbf{d}_1^\ell - \mathbf{b}_{k'}^\ell; R^\ell) - N(\mathbf{d}_3^\ell - \mathbf{b}_{k'}^\ell; R^\ell) \right] \right\}. \end{aligned} \tag{3.15}$$

Extension to maximum call with sequential external barriers

Unlike the two-sided barrier provision where the option is knocked out when the barrier variable hits either H or L, the sequential barrier provision requires the breaching of the two barrier levels at a pre-determined sequential order, say, up then down. For the one-asset case, given the asset price S_1 at the zeroth time, the density function of the asset price S_1^t at time t conditional on non-breaching of the sequential barrier provision (first upstream barrier H then downstream L) is given by (Sidenius, 1998; Li, 1999)

$$\begin{aligned}\phi_{up-down}(x_1,t) &= [\psi_1(x_1-\mu_1 t,t)-\psi_1(x_1-2M_1-\mu_1 t,t)] \\ &\quad - e^{-2\mu_1(M_1-m_1)}[\psi_1(x_1+2(M_1-m_1)-\mu_1 t,t)-\psi_1(x_1-2m_1-\mu_1 t,t)],\end{aligned} \tag{3.16}$$

where $x_1=\dfrac{1}{\sigma_1}\ln\dfrac{S_1^t}{S_1}, \mu_1=\dfrac{r-\frac{\sigma_1^2}{2}}{\sigma_1}, M_1=\dfrac{1}{\sigma_1}\ln\dfrac{H}{S_1}$ and $m_1=\dfrac{1}{\sigma_1}\ln\dfrac{L}{S_1}$.

We consider the multi-asset maximum call option with an external barrier S_1, the terminal payoff of which is gien by $\max(\max(S_2^T,\cdots,S_n^T)-X,0)$. This barrier call option is knocked out only if S_1 hits the up-barrier H first then the down-barrier L afterwards, By following similar derivation procedure as that for the two-sided barrier counterpart, the value of the multi-asset maximum call option with sequential up-then-down barriers is given by

$$\begin{aligned}&V_{up\text{-}down}(S_1,S_2,\cdots,S_n,T)\\ &=\sum_{\ell=2}^{n} S_\ell\left\{e^{\beta_0^\ell}[N(\mathbf{d}_2^\ell-\mathbf{b}_0^\ell;R^\ell)-e^{2\mu_1 M_1+2\beta_{0'}^\ell}N(\mathbf{d}_2^\ell-\mathbf{b}_{0'}^\ell;R^\ell)\right.\\ &\quad\left.-e^{-2\mu_1(M_1-m_1)+2\beta_{-1}^\ell}[N(\mathbf{d}_2^\ell-\mathbf{b}_{-1}^\ell;R^\ell)-e^{2\mu_1(M_1-m_1)+2\beta_{-1'}^\ell}N(\mathbf{d}_2^\ell-\mathbf{b}_{-1'}^\ell;R^\ell)]\right\}\\ &\quad-Xe^{-rT}\left\{N(\mathbf{d}_1^\ell-\mathbf{b}_0^\ell;R^\ell)-e^{2\mu_1 M_1}N(\mathbf{d}_1^\ell-\mathbf{b}_{0'}^\ell;R^\ell)]\right.\\ &\quad\left.-e^{-2\mu_1(M_1-m_1)}[N(\mathbf{d}_1^\ell-\mathbf{b}_{-1}^\ell;R^\ell)-e^{2\mu_1 M_1}N(\mathbf{d}_1^\ell-\mathbf{b}_{-1'}^\ell;R^\ell)]\right\},\end{aligned}$$

where $\mathbf{d}_1^\ell,\mathbf{d}_2^\ell,\beta_0^\ell,\beta_{0'}^\ell,\beta_{-1}^\ell,\beta_{-1'}^\ell,\mathbf{b}_0^\ell,\mathbf{b}_{0'}^\ell,\mathbf{b}_{-1}^\ell$ and $\mathbf{b}_{-1'}^\ell$, are defined in Eqs. (3.14a,b,c,d).

4. MULTI-ASSET OCCUPATION TIME DERIVATIVES

An option is said to be an occupation time derivative if the terminal payoff depends on the terminal asset prices and the occupation time associated with a barrier variable. In this section, we derive the price formulas of several types of occupation time derivatives. We start with the review of some of the results about the one-asset occupation time derivatives and examine how the pricing formulation of the multi-asset models of occupation time derivatives can be inferred from their one-asset counterparts.

4.1 Review on the results for one-asset occupation time derivatives

Let S_1^t be a stochastic variable with the barrier level B. We assume that S_1^t follows the lognormal diffusion process defined in Eq. (3.1). The occupation times of the stochastic variable S_1 below and above the barrier level B from the zeroth time to time t are random variables defined by

$$\tau_B^- = \int_0^t H(B-S_1^u)\,du \tag{4.1a}$$

$$\tau_B^+ = \int_0^t H(S_1^u-B)\,du, \tag{4.1b}$$

respectively, where $H(x)$ is the Heaviside function. The occupation time of the stochastic variable above (below) the barrier level B is the amount of the time that the value of the stochastic variable stays higher (lower) than B. The derivatives of τ_B^- and τ_B^+ are given by

$$d\tau_B^- = H(B - S_1^u)\ dt \quad \text{and} \quad d\tau_B^+ = H(S_1^u - B)\ dt. \tag{4.2}$$

Consider an occupation time derivatives whose terminal payoff function takes the form $F(S_1, \tau_B^-)$. Let $V(S_1, \tau_B^-, t)$ denote the derivative value at time t. Assuming the usual Black-Scholes assumptions, the governing equation for $V(S_1, \tau_B^-, t)$ is given by (Linetsky, 1999)

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma_1^2}{2} S_1^2 \frac{\partial^2 V}{\partial S_1^2} + rS_1 \frac{\partial V}{\partial S_1} + H(B - S_1)\frac{\partial V}{\partial \tau_B^-} - rV = 0, \\ &\qquad\qquad\qquad\qquad t > 0, 0 < S_1 < \infty, \tau_B^- > 0. \end{aligned} \tag{4.3}$$

Note that an additional term $H(B - S_1)\dfrac{\partial V}{\partial \tau_B^-}$ is added in the usual Black-Scholes equation to reflect the dependence of the derivative value on the occupation time state variable τ_B^-. We write $x_1 = \dfrac{1}{\sigma_1} \ln \dfrac{S_1^t}{B}$ and $\xi_1 = \dfrac{1}{\sigma_1} \ln \dfrac{S_1}{B}$, where S_1 and S_1^t are the respective asset prices at the current time (taken to be the zeroth time) and future time t. The transition density function $\psi(x_1, \tau_B^-, t; \xi_1)$ satisfies the following forward Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\frac{\partial^2 \psi}{\partial x_1^2} - \mu_1 \frac{\partial \psi}{\partial x_1} - H(-x_1)\frac{\partial \psi}{\partial \tau_B^-}, \qquad -\infty < x_1 < \infty, \tau_B^- > 0, \tag{4.4}$$

where $\mu_1 = \dfrac{r - \frac{\sigma_1^2}{2}}{\sigma_1}$. The associated initial conditions at $t = 0$ and $\tau_B^- = 0$ are

$$\psi(x_1, \tau_B^-, 0; \xi_1) = \delta(x_1 - \xi_1)\delta(\tau_B^-) \quad \text{and} \quad \psi(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1), \tag{4.5}$$

respectively. Here, $\psi_B(x, t, \xi)$ is the transition density function of the corresponding restricted asset price process without crossing the down barrier B. The condition $\tau_B^- = 0$ is equivalent to the situation where the asset price never breaches the down barrier B. Hence, the value of $\psi(x_1, \tau_B^-, t; \xi_1)$ at $\tau_B^- = 0$ is equal to $\psi_B(x_1, t; \xi_1)$. Also, the initial condition $\psi(x_1, \tau_B^-, 0; \xi_1)$ is derived from the fact S_1^t and τ_B^- start at $t = 0$ with certainty at S_1 and zero value, respectively.

Linetsky (1999) obtained the solution to $\psi(x_1, \tau_B^-, t; \xi_1)$ corresponding to the zero drift case [that is, setting $\mu_1 = 0$ in Eq. (4.4)]. The solution takes different forms in different domains Ω_i, $i = 1, \ldots, 4$, namely,

1. $\Omega_1 = \{(x_1, \xi_1) : \xi_1 \geq 0, x_1 \geq 0 \text{ and } \xi_1 + x_1 > 0\}$

$$\psi = u_1(x_1, \tau_B^-, t; \xi_1) = \int_0^{t-\tau_B^-} \frac{x_1 + \xi_1}{2\pi(t-u)^{3/2}u^{3/2}} \exp\left(-\frac{(x_1 + \xi_1)^2}{2u}\right) du. \tag{4.6a}$$

2. $\Omega_2 = \{(x_1, \xi_1) : \xi_1 \leq 0 \text{ and } x_1 > 0\}$

$$(4.6b) \qquad \psi = u_2(x_1, \tau_B^-, t; \xi_1) = \int_0^{t-\tau_B^-} \frac{x_1\left(1 - \frac{\xi_1^2}{t-u}\right) + \xi_1\left(1 - \frac{x_1^2}{u}\right)}{2\pi(t-u)^{3/2}u^{3/2}} \exp\left(-\frac{x_1^2}{2u} - \frac{\xi_1^2}{2(t-u)}\right) du.$$

3. $\Omega_3 = \{(x_1, \xi_1) : \xi_1 \geq 0, x_1 < 0\}$

$$(4.6c) \qquad \psi = u_3(x_1, \tau_B^-, t; \xi_1) = u_2(-x_1, t - \tau_B^-, t; -\xi_1).$$

4. $\Omega_4 = \{(x_1, \xi_1) : \xi_1 \leq 0, x_1 \geq 0 \text{ and } \xi_1 + x_1 < 0\}$

$$(4.6d) \qquad \psi = u_4(x_1, \tau_B^-, t; \xi_1) = u_1(-x_1, t - \tau_B^-, t; -\xi_1).$$

When $\tau_B^- = 0$ or $\tau_B^- = t$, we have

$$(4.6e) \qquad \psi = \psi_B(x_1, t; \xi_1) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(x_1 - \xi_1)^2}{2t}\right) - \exp\left(-\frac{(x_1 + \xi_1)^2}{2t}\right)\right].$$

Remarks

1. For the non-zero drift case, we can apply Theorem 1 to obtain the corresponding solution for ψ. For example, with $\mu_1 \neq 0, u_1$ becomes

$$(4.7a) \qquad u_1(x_1, \tau_B^-, t; \xi_1, \mu) = Q_1 \int_0^{t-\tau_B^-} \frac{x_1 + \xi_1}{2\pi(t-u)^{3/2}u^{3/2}} \exp\left(-\frac{(x_1 + \xi_1)^2}{2u}\right) du,$$

   where

$$(4.7b) \qquad Q_1 = \exp\left(-\frac{(x_1 - \mu_1 t)^2 - x_1^2}{2t}\right).$$

2. Suppose we count the occupation time starting at t_s with $t_s < 0$, that is, before the current time. The accumulation of occupation time from t_s to the current time is a known quantity since it can be evaluated from the already known realization of the asset price path. The terminal payoff of any occupation time derivative can be modified so that the payoff depends on the occupation time counting from the current time to maturity. Without loss of generality, it suffices to consider those cases where the counting of the occupation time starts at the current time.
3. Successive Laplace transforms on τ_B^- and t are applied to obtain the above solutions for $\psi(x_1, \tau_B^-, t;\ \xi_1)$. The imposition of the initial condition: $\psi(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1)$ seems to lead some complexity in the derivation procedure. Fortunately, $\psi_B(x_1, t; \xi_1)$ does not enter into the equation for the Laplace transform function since the factor $H(-x_1)$ becomes zero when $\tau_B^- = 0$. This is because S_1^t is guaranteed to stay above B when the occupation time τ_B^- is zero.

4.2 Multi-asset occupation time derivatives

We would like to examine how the splitting direction technique can be applied to obtain the price formulas for multi-asset occupation time derivatives. Let S_1^t denote the value of the barrier variable at time t and B be the constant down-barrier level associated with S_1^t. We consider the pricing of multi-asset options whose terminal payoff depends on the terminal asset values $S_2^T, \cdots, S_n^T$, and the terminal value of the occupation time variable. The occupation time variable τ_B^- associated with the barrier variable S_1^t is defined by Eq. (4.1a).

We assume that the asset price processes $S_i^t, i = 1, 2, \cdots, n$, follow the lognormal diffusion processes as defined in Eq. (3.1). Similarly, we define

$$x_j = \frac{1}{\sigma_j} \ln \frac{S_j^t}{S_j}, \quad j = 2, \cdots, n, \tag{4.8}$$

where S_j^t and S_j are the value of asset j at the future time and the current time (taken to be the zeroth time), respectively. The valuation of a multi-asset occupation time derivative requires the determination of the transition density function $\boldsymbol{\psi}(\mathbf{x}, \tau_B^-, t; \xi_1)$ of the joint process of the asset values and the occupation time, where $\mathbf{x}$ denotes the vector $(x_1\ x_2 \cdots x_n)^T$. The forward Fokker-Planck equation that governs $\boldsymbol{\psi}(\mathbf{x}, \tau_B^-, t; \xi_1)$ takes the form

$$\begin{aligned} \frac{\partial \boldsymbol{\psi}}{\partial t} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \boldsymbol{\psi}}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \mu_j \frac{\partial \boldsymbol{\psi}}{\partial x_j} - H(-x_1) \frac{\partial \boldsymbol{\psi}}{\partial \tau_B^-}, \\ & -\infty < x_j < \infty, j = 1, 2, \cdots, n, t > 0, \tau_B^- > 0. \end{aligned} \tag{4.9}$$

Here, ρ_{ij} is the correlation coefficient between dz_i and dz_j, and $\mu_j = \dfrac{r - \frac{\sigma_j^2}{2}}{\sigma_j}, j = 1, 2, \cdots, n$. The initial conditions at $t = 0$ and $\tau_B^- = 0$ are given by

$$\boldsymbol{\psi}(\mathbf{x}, \tau_B^-, 0; \xi_1) = \delta(x_1 - \xi_1)\delta(x_2) \cdots \delta(x_n)\delta(\tau_B^-) \text{ and } \boldsymbol{\psi}(\mathbf{x}, 0, t; \xi_1) = \boldsymbol{\psi}_B(\mathbf{x}, t; \xi_1), \tag{4.10}$$

where $\boldsymbol{\psi}_B(\mathbf{x}, t; \xi_1)$ is the transition density function of the joint process of $S_1^t, S_2^t, \cdots, S_n^t$ with S_1^t staying above B at all times.

The splitting direction technique stated in Theorem 2 can be applied to solve for $\boldsymbol{\psi}(\mathbf{x}, \tau_B^-, t; \xi_1)$. We employ the linear transformation of the independent variables as given in Eq. (2.7), and this leads to the splitting of $\boldsymbol{\psi}(\mathbf{z}, \tau_B^-, t; \xi_1) = \psi_1(z_1, \tau_B^-, t; \xi_1)\psi_{n-1}(z_2, \cdots, z_n, t; \xi_1)$. The procedure is motivated by observing that the function in the boundary condition in Eq. (4.10) can be splitted by the same linear transformation, i.e. $\psi_B(\mathbf{x}, t; \xi_1) = \psi_B(z_1, t; \xi_1)\psi_{n-1}(\hat{\mathbf{z}}, t)$. The governing equations and auxiliary conditions for $\psi_1(z_1, \tau_B^-, t; \xi_1)$ and $\psi_{n-1}(z_2, \cdots, z_n, t; \xi_1)$ are given by

$$\frac{\partial \psi_1}{\partial t} = \frac{1}{2} \frac{\partial^2 \psi_1}{\partial z_1^2} - \mu_1 \frac{\partial \psi_1}{\partial z_1} - H(-z_1) \frac{\partial \psi_1}{\partial \tau_B^-}, \quad t > 0, \tau_B^- > 0, -\infty < z_1 < \infty, \tag{4.11a}$$

$$\psi_1(x_1, \tau_B^-, 0; \xi_1) = \delta(x_1 - \xi_1)\delta(\tau_B^-), \qquad \psi_1(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1),$$

$$\frac{\partial \psi_{n-1}}{\partial t} = \frac{1}{2} \sum_{i=2}^{n} \sum_{j=2}^{n} \widetilde{\rho}_{ij} \frac{\partial^2 \psi_{n-1}}{\partial z_i \partial z_j} - \sum_{j=2}^{n} \widetilde{\mu}_j \frac{\partial \psi_{n-1}}{\partial z_j}, \quad t > 0, -\infty < z_j < \infty, j = 2, \cdots, n, \tag{4.11b}$$

$$\psi_{n-1}(\hat{\mathbf{z}}, \tau_B^-, t; \xi_1) = \delta(\hat{\mathbf{z}}),$$

respectively. The correlation coefficients $\widetilde{\rho}_{ij}$ and the drift parameters $\widetilde{\mu}_j$ are defined in Eq. (2.10b).

We let R^{n-1} denote $\{(x_2,\dots,x_n): -\infty < x_j < \infty, j=2,\dots,n\}$ and define the domains Ω_i^n, $i=1,\dots,4$, to be $\Omega_i^n = \Omega_i \times R^{n-1}$, where Ω_i's are defined in Eqs. (4.6a-d). The solution to the transition density function $\psi(\mathbf{x},\tau_B^-,t;\xi_1)$ in different domain Ω_i^n, $i=1,\dots,4$, are summarized in Theorem 4.

Theorem 4

Let ψ_k^n be the solution to the joint transition density function $\psi(\mathbf{x},\tau_B^-,t;\xi_1)$ in the domain Ω_i^n, $i=1,\dots,4$. We have

$$\psi(\mathbf{x},\tau_B^-,t;\xi_1) = \mathbf{1}_{\{\Omega_1^n\}}\psi_1^n + \mathbf{1}_{\{\Omega_2^n\}}\psi_2^n + \mathbf{1}_{\{\Omega_3^n\}}\psi_3^n + \mathbf{1}_{\{\Omega_4^n\}}\psi_4^n + [\delta(\tau_B^-) + \delta(\tau_B^- - t)]\psi_5^n. \tag{4.12}$$

where

(4.13)

$$\begin{aligned}
\psi_1^n =& \sqrt{\frac{t}{2\pi}}\psi_n(\mathbf{x}-\xi_1\boldsymbol{e}_1-\boldsymbol{\mu}t;R,t)\int_0^{t-\tau_B^-}\frac{x_1+\xi_1}{(t-u)^{3/2}u^{3/2}}\exp\left(\frac{(x_1-\xi_1)^2}{2t}-\frac{(x_1+\xi_1)^2}{2u}\right)du,\\
\psi_2^n =& \sqrt{\frac{t}{2\pi}}\psi_n(\mathbf{x}-\xi_1\boldsymbol{e}_1-\boldsymbol{\mu}t;R,t)\\
&\times\int_0^{t-\tau_B^-}\frac{[x_1(1-\frac{\xi_1^2}{t-u})+\xi_1(1+\frac{x_1^2}{u})]}{(t-u)^{3/2}u^{3/2}}\exp\left(\frac{(x_1-\xi_1)^2}{2t}-\frac{x_1^2}{2u}-\frac{{\xi_1}^2}{2(t-u)}\right)du,\\
\psi_3^n&(x_1,x_2,\cdots,x_n,\tau_B^-,t;\xi_1) = \psi_2^n(-x_1,x_2,\cdots,x_n,t-\tau_B^-,t;-\xi_1),\\
\psi_4^n&(x_1,x_2,\cdots,x_n,\tau_B^-,t;\xi_1) = \psi_1^n(-x_1,x_2,\cdots,x_n,t-\tau_B^-,t;-\xi_1),\\
\psi_5^n &= \psi_n(\mathbf{x}-\xi_1\boldsymbol{e}_1-\boldsymbol{\mu}t,t;R) - e^{2u_1\xi_1}\psi_n(\mathbf{x}-\xi_1\boldsymbol{e}_1-\boldsymbol{\mu}t-2\xi_1R\boldsymbol{e}_1,t;R),
\end{aligned}$$

where $\psi_n(\mathbf{x},t;R)$ is the fundamental solution defined in Eq. (2.11).

Let $V(S_1,\dots,S_n,T)$ denote the value of the n-asset occupation time derivative with the down barrier B on S_1^t and payoff function $P(\mathbf{x},\tau_B^-)$. The price function $V(S_1,\dots,S_n,T)$ is given by

(i) $S_1 \ge B$

$$\begin{aligned}
&V(S_1,\dots,S_n,T)\\
&= e^{-rT}\left\{\int_{R^{n-1}}\int_0^\infty P(\mathbf{x},0)\psi_5^n\ d\mathbf{x} + \int_0^T\int_{R^{n-1}}\int_0^\infty P(\mathbf{x},\tau_B^-)\psi_1^n\ d\mathbf{x}d\tau_B^-\right.\\
&\qquad\left. + \int_0^T\int_{R^{n-1}}\int_{-\infty}^0 P(\mathbf{x},\tau_B^-)\psi_3^n\ d\mathbf{x}d\tau_B^-\right\};
\end{aligned}\tag{4.14a}$$

(ii) $S_1 \le B$

$$\begin{aligned}
&V(S_1,\dots,S_n,T)\\
&= e^{-rT}\left\{\int_{R^{n-1}}\int_{-\infty}^0 P(\mathbf{x},T)\psi_5^n\ d\mathbf{x} + \int_0^T\int_{R^{n-1}}\int_0^\infty P(\mathbf{x},\tau_B^-)\psi_2^n\ d\mathbf{x}d\tau_B^-\right.\\
&\qquad\left. + \int_0^T\int_{R^{n-1}}\int_\infty^0 P(\mathbf{x},\tau_B^-)\psi_4^n\ d\mathbf{x}d\tau_B^-\right\}.
\end{aligned}\tag{4.14b}$$

In particular, we can apply the above formula to compute the price of the proportional step options, simple step options and delayed barrier options (or called the cumulative Parisian options). They are occupation time derivatives having payoff of the seperable form: $f(\tau_B^-)G(\mathbf{x})$.

1. Proportional step option:

$$f(\tau_B^-) = e^{-s\tau_B^-},$$

   where s is called the killing rate.

2. Simple step option:

$$f(\tau_B^-) = \max(1 - s\tau_B^-, 0).$$

3. Delayed barrier option:

$$f(\tau_B^-) = \mathbf{1}_{\{\tau_B^- < \alpha T\}},$$

   where α is a parameter satisfying $0 < \alpha < 1$.

Theorem 5

Consider the maximum call option with S_1 as the external barrier variable and the associated occupation time τ_B^-. Suppose the terminal payoff takes the separable form: $f(\tau_B^-)\max(\max(S_2, \cdots, S_n)$ $-K, 0)$. The corresponding price formula of this n-asset occupation time derivative with the maximum call payoff is found to be

(i) $S_1 \geq B$

$$\begin{aligned}
&V(S_1, \ldots, S_n, T) \\
&= f(0)\ DOC_{max}^{n-1}(S_1, \ldots, S_n, T) \\
&+ \int_0^T F(T - \tau_B^-)\Bigg[\int_0^\infty \Bigg(\sum_{k=2}^n S_k \exp(-\rho_{1k}\sigma_k(\xi_1 - x_1 + \mu_1 T))N(\boldsymbol{g}_{k,1} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}\hat{R}_k; \hat{R}_k) \\
&\qquad - \sum_{k=2}^n Ke^{-rT}N(\boldsymbol{g}_{k,2} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}; \hat{R}_k)\Bigg)\frac{\partial u_1}{\partial \tau_B^-}(x_1, T - \tau_B^-, T; \xi_1)\ dx_1 \\
&+ \int_{-\infty}^0 \Bigg(\sum_{k=2}^n S_k \exp(-\rho_{1k}\sigma_k(\xi_1 - x_1 + \mu_1 T))N(\boldsymbol{g}_{k,1} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}\hat{R}_k; \hat{R}_k) \\
&\qquad - \sum_{k=2}^n Ke^{-rT}N(\boldsymbol{g}_{k,2} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}; \hat{R}_k)\Bigg)\frac{\partial u_3}{\partial \tau_B^-}(x_1, T - \tau_B^-, T; \xi_1) d\xi_1\Bigg]\ d\tau_B^-
\end{aligned} \tag{4.15a}$$

(ii) $\quad S_1 \leq B$

$$
\begin{aligned}
& V(S_1,\ldots,S_n,T) \\
&= \int_0^T F(T-\tau_B^-)\Bigg[\int_0^\infty \Bigg(\sum_{k=2}^n S_k \exp(-\rho_{1k}\sigma_k(\xi_1 - x_1 + \mu_1 T)) N(\boldsymbol{g}_{k,1} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}\hat{R}_k;\hat{R}_k) \\
&\qquad - \sum_{k=2}^n K e^{-rT} N(\boldsymbol{g}_{k,2} - C_k\hat{\boldsymbol{\mu}}\sqrt{T};\hat{R}_k)\Bigg) \frac{\partial u_2}{\partial \tau_B^-}(x_1, T-\tau_B^-, T;\xi_1)\, dx_1 \quad (4.15b) \\
&\quad + \int_{-\infty}^0 \Bigg(\sum_{k=2}^n S_k \exp(-\rho_{1k}\sigma_k(\xi_1 - x_1 + \mu_1 T)) N(\boldsymbol{g}_{k,1} - C_k\hat{\boldsymbol{\mu}}\sqrt{T}\hat{R}_k;\hat{R}_k) \\
&\qquad - \sum_{k=2}^n K e^{-rT} N(\boldsymbol{g}_{k,2} - C_k\hat{\boldsymbol{\mu}}\sqrt{T};\hat{R}_k)\Bigg) \frac{\partial u_4}{\partial \tau_B^-}(x_1, T-\tau_B^-, T;\xi_1)\, dx_1\Bigg] d\tau_B^-
\end{aligned}
$$

where $DOC_{max}^{n-1}(S_1,\ldots,S_n,T)$ denotes the price formula of the corresponding down-and-out maximum call option, and the functions $u_j, j = 1,\ldots,4$ are given in Eq.(4.6a-d). The other paramaters are defined as follows:

$$
\boldsymbol{g}_{k,2} = (g_2^k \ \cdots \ g_n^k)^T, \quad \boldsymbol{g}_{k,1} = \boldsymbol{g}_{k,2} + \sigma_k\sqrt{T}\hat{R}_k\mathbf{e}_k, \quad \hat{R}_k = C_k\hat{R}C_k^T, \quad \hat{\boldsymbol{\mu}} = (\mu_2\cdots\mu_n)^T,
$$

$$
(4.16) \qquad g_j^k = \begin{cases} \dfrac{1}{\sigma_k\sqrt{k}} \ln\dfrac{S_k}{K} - \rho_{1k}(x_1 - \xi_1), & j = k \\[2ex] \dfrac{1}{\sigma_{jk}\sqrt{\tau}} \ln\dfrac{S_k}{S_j} + \left(\dfrac{\rho_{1k}\sigma_k - \rho_{1j}\sigma_j}{\sigma_{jk}}\right)(x_1 - \xi_1), & \text{otherwise} \end{cases};
$$

$$
F(T-\tau_B^-) = \int_0^{T-\tau_B^-} f(u)\, du.
$$

The entries in the matrix C_k are given by

$$
(4.17) \qquad C_{ij}^k = \begin{cases} -\sqrt{1-\rho_{1k}^2}, & i = j = k, \\[1ex] \dfrac{\sigma_j}{\sigma_{jk}}\sqrt{1-\rho_{1j}^2}, & i = j \neq k, \\[1ex] -\dfrac{\sigma_k}{\sigma_{jk}}\sqrt{1-\rho_{1k}^2}, & j = k, \\[1ex] 0, & \text{otherwise.} \end{cases}
$$

Remarks

For different types of occupation time derivatives with separable terminal payoff function, the function $F(T-\tau_B^-)$ takes different forms.

1. Proportional step option

$$
F(T-\tau_B^-) = \int_0^{T-\tau_B^-} e^{-su}\, du = \frac{1-e^{-s(T-\tau_B^-)}}{s}.
$$

2. Delayed barrier option

$$F(T-\tau_B^-) = \int_0^{T-\tau_B^-} \mathbf{1}_{u<\alpha T}\, du$$
$$= \begin{cases} \alpha T, & 0 \le \tau_B^- \le (1-\alpha)T \\ T-\tau_B^-, & (1-\alpha)T < \tau_B^- \le T \end{cases}.$$

3. Simple step option

$$F(T-\tau_B^-) = \int_0^{T-\tau_B^-} \max(1-su, 0)\, du$$
$$= \begin{cases} \dfrac{1}{2s}, & 0 \le \tau_B^- \le T - \dfrac{1}{s} \\ (T-\tau_B^-)\left[1 - \dfrac{s}{2}(T-\tau_B^-)\right], & T - \dfrac{1}{s} < \tau_B^- \le T \end{cases}.$$

5. CONCLUSION

Since option prices are given by the discounted expectation of the terminal payoff in the risk neutral world, the derivation of the analytical price formulas of exotic option models amounts to the analytical evaluation of expectation integrals. A typical form of the integrand in an expectation integral is given by the product of the transition density function and the terminal payoff function. For multi-asset barrier-type options and occupation time derivatives, the derivation of the associated density function has been known to be mathematically challenging, due primarily to the presence of the cross-diffusion terms in the Fokker-Planck equation. In this paper, we develop the splitting direction technique which leads to a systematic derivation approach to find the density functions of multi-asset option models from the extension of their one-asset counterparts. Analytic price formulas of multi-asset options with external two-sided barriers and sequential barriers, multi-asset step options and delayed barrier options are obtained in their most succinct forms.

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Appendix A - proof of Theorem 2

Let $\nabla_{\mathbf{x}} = (\partial_{x_1} \cdots \partial_{x_n})^T$ and let J denote the matrix representing the linear transformation between $\mathbf{x} = (x_1 \cdots x_n)^T$ and $z = (z_1 \cdots z_n)^T$ as defined in Eq. (2.7), that is, $\mathbf{z} = J\mathbf{x}$. We have $J = \dfrac{\partial(z_1, \cdots, z_n)}{\partial(x_1, \cdots, x_n)}$, so $\nabla_{\mathbf{x}} = J\nabla_{\mathbf{z}}$. Also, we write $\boldsymbol{\mu} = (\mu_1 \cdots \mu_n)^T$. Now, Eq. (2.6) can be expressed as

$$\begin{aligned}\frac{\partial \phi_n}{\partial t} &= \frac{1}{2}\nabla_{\mathbf{x}}^T R \nabla_{\mathbf{x}}\phi_n - \boldsymbol{\mu}^T \nabla_{\mathbf{x}}\phi_n \\ &= \frac{1}{2}\nabla_{\mathbf{z}}^T (J^T R J)\nabla_{\mathbf{z}}\phi_n - (J^T\boldsymbol{\mu})^T \nabla_{\mathbf{z}}\phi_n.\end{aligned} \tag{A.1}$$

Let $\widehat{\rho}_{ij}$ denote the $(i,j)^{\text{th}}$ entry of $J^T RJ$ and $\widehat{\mu}_j$ denote the j^{th} entry of $J^T\boldsymbol{\mu}$. We observe that

$$\widehat{\rho}_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{if } i = 1, j \neq 1 \text{ or } i \neq 1, j = 1 \\ \widetilde{\rho}_{ij} & \text{if } i \neq 1, j \neq 1 \end{cases} \tag{A.2}$$

$$\widehat{\mu}_j = \begin{cases} \mu_1 & j = 1 \\ \widetilde{\mu}_j & j \neq 1 \end{cases},$$

where $\widetilde{\rho}_{ij}$ and $\widetilde{\mu}_j$ are defined in Eq. (2.10b). In terms of $z_1, \cdots, z_n$, Eq. (2.7) can be expressed as

$$\frac{\partial \phi_n}{\partial t} = \left[\frac{1}{2}\frac{\partial^2 \phi_n}{\partial z_1^2} - \mu_1 \frac{\partial \phi_n}{\partial z_1}\right] + \left[\frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\widetilde{\rho}_{ij}\frac{\partial^2 \phi_n}{\partial z_i \partial z_j} - \sum_{j=2}^{n}\widetilde{\mu}_j \frac{\partial \phi_n}{\partial z_j}\right]. \tag{A.3}$$

Suppose we decompose ϕ_n into the form

$$\phi_n(z_1, z_2, \cdots, z_n, t) = \phi_1(z_1, t)\phi_{n-1}(z_2, \cdots, z_n, t), \tag{A.4}$$

then ϕ_1 and ϕ_{n-1} satisfy Eq. (2.9) and Eq. (2.10a), respectively.

Appendix B - proof of Eq. (2.12)

Let $\psi_1(z_1, t)$ and $\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R})$ denote the respective fundamental solutions to the following equations

$$\frac{\partial \psi_1}{\partial t} = \frac{1}{2}\frac{\partial^2 \psi_1}{\partial z_1^2}, \quad t > 0, -\infty < z_1 < \infty, \tag{B.1}$$

$$\frac{\partial \psi_{n-1}}{\partial t} = \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\widetilde{\rho}_{ij}\frac{\partial^2 \psi_{n-1}}{\partial z_i \partial z_j}, \quad t > 0, -\infty < z_j < \infty, j = 2, \cdots, n. \tag{B.2}$$

Here, Eqs. (B.1) and (B.2) are seen to correspond to Eq. (2.9) and Eq. (2.10a) with zero drift terms, respectively. Similar to relation (2.8), the fundamental solutions $\psi_n(x, t; R), \psi_1(z_1, t)$ and $\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R})$ are related by

$$\psi_n(\mathbf{x}, t; R) = \psi_1(z_1, t)\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R}) \det J. \tag{B.3}$$

The Jacobian $\det J = \left|\dfrac{\partial(z_1,\cdots,z_n)}{\partial(x_1,\cdots,x_n)}\right|$ is included due to the change of the independent variable from $\mathbf{x}$ to $\mathbf{z}$.

Suppose we apply a shifting transformation on $\mathbf{x}$: $\mathbf{x}_{new} = \mathbf{x}_{old} - \alpha R\boldsymbol{e}_1$, where α is any scalar, then $\mathbf{z}_{new} = J\mathbf{x}_{new} = J(\mathbf{x}_{old} - \alpha R\boldsymbol{e}_1) = \mathbf{z}_{old} - \alpha\boldsymbol{e}_1$. We observe that $\mathbf{z}_{new}$ is obtained from $\mathbf{z}_{old}$ by changing only the first component z_1 to $z_1 - \alpha$ while keeping all the other $n-1$ components in $\mathbf{z}_{old}$ unchanged. Accordingly, the relation for the fundamental solutions as stated in Eq. (B.3) is modified as

$$(B.4) \qquad \psi_n(\mathbf{x} - \alpha R\boldsymbol{e}_1, t; R) = \psi_1(z_1 - \alpha, t)\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R}) \det J,$$

where $\widetilde{\mathbf{z}}$ and the Jacobian $\det J$ remain unchanged under this shifting transformation on $\mathbf{x}$.

Appendix C - proof of Proposition 3

With no loss of generality, we prove the proposition for the case $k = 1$. We define a new vector of Brownian motions $\boldsymbol{Z}$ by $\boldsymbol{Z} = J\boldsymbol{X}$, where J is the matrix representing the linear transformation of variables defined in Eq. (2.7). We have

$$(C.1) \qquad d\mathbf{P}_n = [\psi_1(z_1, t)dz_1)][\psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R})d\widetilde{\mathbf{z}}] = dP_1 d\mathbf{P}_{n-1},$$

where $\psi_1(z_1, t), \psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R}), \widetilde{\mathbf{z}}$ and $\widetilde{R}$ are defined in the same manner as those in Eq. (2.12). Note that $Z_1(t)$ is a Brownian motion under the probability measure P_1. Next, we define

$$(C2a) \qquad W_1(t) = Z_1(t) + \lambda t$$

$$(C.2b) \qquad L_1(t) = \exp\left(-\lambda Z_1(t) - \frac{\lambda^2 t}{2}\right)$$

where λ is any constant. By the Girsanov Theorem, we deduce that $W_1(t)$ is a Brownian motion under the new probability measure Q_1 that satisfies the Radon-Nikodym derivative $\dfrac{dQ_1}{dP_1} = L_1(t)$. Accordingly, we define the probability measure Q_n^1 by the property

$$(C.3) \qquad dQ_n^1 = dQ_1 d\mathbf{P}_{n-1} = L_1(t)d\mathbf{P}_n,$$

which then gives the result in Eq. (2.14). On the other hand, dQ_n^1 observes

$$\begin{aligned}
d\mathbf{Q}_n^1 &= dQ_1 d\mathbf{P}_{n-1} \\
&= \psi_1(W_1(t), t)dW_1 \; \psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R})d\widetilde{\mathbf{z}} \\
&= \psi_1(Z_1(t) + \lambda t, t) \; \psi_{n-1}(\widetilde{\mathbf{z}}, t; \widetilde{R})d\mathbf{z} \\
&= \psi_n(\boldsymbol{X} + \lambda t R\boldsymbol{e}_1, t; R)d\mathbf{z} \quad \text{(by Corollary of Theorem 2).}
\end{aligned}$$

Hence, $\boldsymbol{X} + \lambda t R\boldsymbol{e}_1$ is a vector of Brownian motions under $\mathbf{Q}_n^1$.

Appendix D - proof of Theorem 5

Suppose the terminal payoff function of the multi-asset occupation derivative is in the separable form, i.e. $P(\mathbf{x}, \tau_B^-) = f(\tau_B^-)G(\mathbf{x})$. The integrals appearing in Eqs. (4.14a,b) become

(i) $S_1 \geq B$

$$(D.1a) \qquad V = f(0)DOC + \int_0^T F(T-\tau_B)\left[\int_0^\infty \int_{R^{n-1}} G(\mathbf{x})\frac{\partial \psi_1^n(\mathbf{x}, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-}\, d\mathbf{x} + \int_{-\infty}^0 \int_{R^{n-1}} G(\mathbf{x})\frac{\partial \psi_2^n(\mathbf{x}, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-} d\mathbf{x}\right] d\tau_B^-,$$

(ii) $S_1 \leq B$

$$(D.1b) \qquad V = \int_0^T F(T-\tau_B^-)\left[\int_0^\infty \int_{R^{n-1}} G(\mathbf{x})\frac{\partial \psi_3^n(\mathbf{x}, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-}\, d\mathbf{x} + \int_{-\infty}^0 \int_{R^{n-1}} G(\mathbf{x})\frac{\partial \psi_4^n(\mathbf{x}, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-} d\mathbf{x}\right] d\tau_B^-$$

In particular, we consider the maximum call option with the external barrier variable S_1 and $G(\mathbf{x}) = \max(\max(S_2e^{\sigma_2 x_2}, \dots, S_n e^{\sigma_n x_n}) - K, 0)$. As an illustration, we only show the evaluation of the following typical integral

$$I = \int_0^T F(T-\tau_B^-) \int_{R^{n-1}} \int_0^\infty G(\mathbf{x})\frac{\partial \psi_1^n(\mathbf{x}, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-}\, d\mathbf{x} d\tau_B^-$$

$$(D.2a) \qquad = \int_0^T F(T-\tau_B^-) \int_0^\infty \sum_{k=2}^n I_k \frac{\partial u_1(x_1, T-\tau_B^-, T; \xi_1)}{\partial \tau_B^-}\, dx_1\, d\tau_B^-;$$

where

$$(D.2b) \qquad I_k = \int_{\mathcal{D}_k} \left[S_k e^{\sigma_k (z_k\sqrt{1-\rho_{1k}^2} + \rho_{1k}(x_1-\xi_1))} - K\right] \phi(\hat{\mathbf{z}} - \hat{\boldsymbol{\mu}}T, T; \hat{R})\, d\hat{\mathbf{z}}.$$

Here $\hat{\mathbf{z}} = (z_2 \cdots z_n)^T$ and $\hat{\boldsymbol{\mu}} = (\hat{\mu}_2 \cdots \hat{\mu}_n)^T$ are in R^{n-1}, $\hat{\mu}_j = \frac{\mu_j - \rho_{1,j}\mu_1}{\sqrt{1-\rho_{1j}^2}}$ and $\mathcal{D}_k$ is the integration region where S_j^T is the maximum among $S_2^T, \dots, S_n^T$. To facilitate the integration, we apply the following transformation of variables:

$$(D.3) \qquad \eta_j^k = \begin{cases} -z_k\sqrt{1-\rho_{1k}^2}, & j = k \\ \dfrac{\sigma_j\sqrt{1-\rho_{ij}^2}}{\sigma_{jk}}\left(z_j - z_k\dfrac{\sigma_k}{\sigma_j}\sqrt{\dfrac{1-\rho_{1k}^2}{1-\rho_{ij}^2}}\right), & \text{otherwise.} \end{cases}$$

Equivalently, the above transformation can be written as $\boldsymbol{\eta}^k = C_k\hat{\mathbf{z}}$, where C_k is presented in Eq. (4.17). The integration domain $\mathcal{D}_k$ becomes

$$\mathcal{D}_k = \left\{(\eta_2^k, \dots, \eta_n^k) \mid \eta_k^k < \frac{1}{\sigma_k}\ln\frac{S_k}{K} - \rho_{1k}(x_1-\xi_1), \; \eta_j^k < \frac{1}{\sigma_{jk}}\ln\frac{S_k}{S_j} + (x_1-\xi_1)\left(\frac{\rho_{1k}\sigma_k - \rho_{1j}\sigma_j}{\sigma_{jk}}\right), j = 2, \dots, n \text{ and } j \neq k\right\}.$$

By the generalized Girsanov theorem in Proposition 3, we obtain

$$(D.3) \quad I_k = \int_{\mathcal{D}_k} [S_k \exp(-\sigma_k \eta_k^k + \rho_{1k}\sigma_k(x_1 - \xi_1)) - K]\phi(\boldsymbol{\eta}^k - C_k \hat{\boldsymbol{\mu}} T, T; \hat{R}_k)\ d\boldsymbol{\eta}^k$$
$$= S_k e^{(r-q_k)T - \rho_{1k}(x_1 - \xi_1 + \mu_1 T)} N(\boldsymbol{g}_{k,1} - C_k \hat{\boldsymbol{\mu}}\sqrt{T}; \hat{R}_k) - KN(\boldsymbol{g}_{k,2} - C_k \hat{\boldsymbol{\mu}}\sqrt{T}; \hat{R}_k).$$

The other terms can be calculated in a similar manner. Hence, the result in Theorem 5 is obtained.