Optimal shouting policies of options with shouting rights

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Date of completion: 20 January, 2001

Abstract. The shouting (resettable) right embedded in an option contract is defined to be the privilege given to the option holder to reset certain terms in the contract according to specified rules at the moment of shouting, where the time to shout is chosen by the holder. This paper develops the framework of analyzing the optimal shouting policies to be adopted by the holder of an option with single and multiple shouting rights. It is most interesting to observe that the optimal shouting boundary depends on the relative values of the riskless interest rate and dividend yield. The monotonicity properties and the asymptotic behaviors at limiting zero value and infinite value of time to expiry associated with the shouting boundaries are examined. For the shout floor with single and multiple shouting rights, we obtain an analytic representation of the price function.

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1. Introduction

The acute competitions in the markets prompt financial engineers to design option contracts with more exotic features. One feature that may be embedded in an option is the right given to the holder to reset certain contract terms according to specified rules during the life of the option contract. One simple example is the *resettable put option*, where the strike price is reset to be the prevailing asset price at the moment chosen by the holder. The moment to reset is often called the shouting moment. Let X denote the original strike price set at initiation of the option, S_t and S_T denote the asset price at the shouting instant t and maturity date T, respectively. The payoff of the resettable put option is given by $\max(X - S_T, 0)$ if no shouting occurs throughout the option's life, and the payoff is modified to $\max(S_t - S_T, 0)$ if shouting occurs at time t before the maturity date T. Upon shouting, the resettable put option effectively converts into an at-the-money put option. From the nature of the payoff, it is obvious that the holder should possibly shout only for $S_t > X$ so that an increase in the terminal payoff is resulted after shouting.

Another example is the *shout floor* where the holder can shout at any time t during the life of the contract to install a floor on the return, with the floor value set at the prevailing asset price S_t at the shouting moment. The right to install the floor is coined the term "shout floor" by Cheuk and Vorst (1997). The terminal payoff of the shout floor is seen to be $\max(S_t - S_T, 0)$ if shouting occurs, but becomes zero if otherwise. Hence, the right to install the floor at the prevailing asset value upon shouting is equivalent to the right given to the holder to choose when to receive an at-the-money put option.

The most original options with the shout feature resemble the ladder options, except that the ladder corresponding to the shout feature is not pre-determined. This class of options were first called "shout options" by Thomas (1993). Consider the shout option with the call payoff, its terminal payoff is given by $\max(S_T - X, S_t - X)$ if shouting occurs at time t, and stays at $\max(S_T - X, 0)$ if otherwise. Here, the "effective ladder" in a shout call option is S_t, but the ladder is chosen by the holder. Again, the holder should optimally choose to shout only for $S_t > X$.

There exist a wide range of financial instruments with embedded shout features. Gray and Whaley (1999) analyzed the resettable feature in the Geared Equity Investment offered by Macquarie Bank. Brenner *et al.* (2000) examined the impact of resetting the terms of previously-issued executive stock options on firm performance. Windcliff *et al.* (2000) analyzed the Canadian segregated funds with multiple reset rights on guaranteed level and maturity date. Jaillet *et al.* (1998) studied a special form of shout feature (swing option) that appears in some energy derivative contracts. Windcliff *et al.* (1999) proposed a range of numerical algorithms for pricing shout options with non-constant volatility in the underlying asset price process.

Similar to American options with the early exercise right, the pricing of options with the shouting right leads to free boundary value problems. One would expect that the analysis framework and the analytic techniques that have been developed for pricing American options can be employed to study the pricing models of options with the shout feature.

In this paper, we develop an analytic framework to analyze the optimal shouting policies for options with single and multiple shouting rights. The paper is organized as follows. In Section 2, we examine the relationship between the one-shout resettable put option and the one-shout shout call option, and formulate the pricing models for various types of options with the shout feature. Section 3 presents the analytic derivation of the price function of the one-shout shout floor. In Section 4, we explore the characterization of the optimal shouting boundary $S_1^*(\tau)$ of the one-shout resettable put option under different conditions on the relative values of the riskless interest rate and the dividend yield. In particular, the asymptotic behaviors of $S_1^*(\tau)$ at $\tau \to 0^+$ and $\tau \to \infty$ are examined. Other properties of $S_1^*(\tau)$ are also discussed. We also give the details on the numerical procedure to compute $S_1^*(\tau)$. The extension of the analysis of the optimal shouting policies to options with multiple shouting rights is performed in Section 5. The paper ends with conclusive remarks in the last section.

2. Formulation of the pricing models

We follow the usual Black-Scholes assumptions in the pricing framework for options with the shout feature. In the risk neutral world, the stochastic process for the asset price S is assumed to follow the lognormal diffusion process

$$\frac{dS}{S} = (r - q)dt + \sigma dZ, \tag{2.1}$$

where r and q are the constant riskless interest rate and dividend yield, respectively, σ is the constant volatility and dZ is the standard Wiener process.

2.1 Relation between the resettable put option and the shout call option

Consider the portfolio of holding an one-shout shout call option and shorting a forward contract with the delivery price same as the strike price of the shout call option. Both derivatives are assumed to have the same initiation date and maturity date T. The terminal payoff of this portfolio is seen to be (i) $\max(S_T - X, 0) - (S_T - X) = \max(X - S_T, 0)$ if there is no shout throughout the whole life of the contracts, and (ii) $\max(S_T - X, S_t - X) - (S_T - X) = \max(S_t - S_T, 0)$ if the holder shouts at time t prior to maturity. Here, S_t and S_T denote the asset price at the shouting instant t and maturity date T, respectively. This payoff structure resembles that of the one-shout resettable put option, where the optimal shouting time to reset the strike is chosen by the holder.

Let $V_1(S, \tau)$ and $U_1(S, \tau)$ denote the price of the one-shout resettable put option and the one-shout shout call option, respectively, where τ is the time to expiry. Since the shout call option can be replicated by the combination of the resettable put option and the corresponding forward contract, both options should share the same optimal shouting policy. In addition, the relation between $V_1(S, \tau)$ and $U_1(S, \tau)$ is given by

$$U_1(S, \tau) = V_1(S, \tau) + Se^{-q\tau} - Xe^{-r\tau}. \tag{2.2}$$

In the remaining of this paper, we will concentrate our discussion on the resettable put option and the shout floor since the optimal shouting policy of the shout call option is exactly the same as that of the corresponding resettable put option.

2.2 Formulation as free boundary value problems

For either the one-shout resettable put option or the one-shout shout floor, the option becomes an at-the-money put option upon shouting. The price function of this at-the-money put option is seen to be linearly homogeneous in S and takes the form $SP_1(\tau)$. By setting the strike price to be the current asset price in the Black-Scholes vanilla put option price formula, we obtain

$$P_1(\tau) = e^{-r\tau} N(-d_2) - e^{-q\tau} N(-d_1), \tag{2.3}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi, \quad d_1 = \frac{r - q + \frac{\sigma^2}{2}}{\sigma}\sqrt{\tau} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau}. \tag{2.4}$$

The pricing model of the resettable put option or the shout floor leads to a free boundary value problem. The linear complementarity formulation of the pricing function $V(S,\tau)$ is given by

$$\begin{aligned} &\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV \geq 0, \quad V(S,\tau) \geq SP_1(\tau), \\ &\left[\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV\right][V - SP_1(\tau)] = 0, \\ &V(S,0) = \begin{cases} \max(X-S,0), & \text{resettable put} \\ 0, & \text{shout floor} \end{cases}. \end{aligned} \tag{2.5}$$

Note that the formulations of the resettable put and the shout floor differ only in the terminal payoff. The critical shouting boundary, denoted by $S_1^*(\tau)$, is not known *a priori* but has to be solved in the solution procedure of the free boundary value problem. Alternatively, we may formulate the pricing model as

$$\begin{aligned} &\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV = 0 \quad \text{for} \quad 0 < S < S_1^*, \tau > 0, \\ &V(0,\tau) = Xe^{-r\tau}, V(S_1^*,\tau) = S_1^* P_1(\tau), \\ &\frac{\partial V}{\partial S}(S_1^*,\tau) = P_1(\tau), \\ &V(S,0) = \begin{cases} \max(X-S,0), & \text{resettable put} \\ 0, & \text{shout floor} \end{cases}. \end{aligned} \tag{2.6}$$

Note that the option price function $V(S,\tau)$ observes the smooth pasting (or "high contact") conditions, that is, continuity of the option value and delta across the optimal shouting boundary $S_1^*(\tau)$. Since the conversion of the option into an at-the-money put option at the critical asset price $S_1^*(\tau)$ is self-financing, the continuity of the option value at $S_1^*(\tau)$ then follows. The validity of the smooth pasting condition can be deduced by following the same argument as that used by Merton (1973) for the American option model.

2.3 Properties of $P_1(\tau)$

The following results on the derivative of $e^{q\tau}P_1(\tau)$ are crucial for the derivation of the pricing formula of the one-shout shout floor as well as the analysis of the optimal shouting boundary for the one-shout resettable put option.

Lemma 2.1 For the function $P_1(\tau)$ defined in Eq. (2.3), the derivative of $e^{q\tau}P_1(\tau)$ observes the following properties.

(i) If $r \leq q$, then

$$\frac{d}{d\tau}[e^{q\tau}P_1(\tau)] > 0 \quad \text{for} \quad \tau \in (0,\infty). \tag{2.7}$$

(ii) If $r > q$, there exists a unique critical value $\tau_1^* \in (0, \infty)$ such that

$$\frac{d}{d\tau}\left[e^{q\tau} P_1(\tau)\right]\Big|_{\tau=\tau_1^*} = 0, \tag{2.8a}$$

and

$$\frac{d}{d\tau}\left[e^{q\tau} P_1(\tau)\right] > 0 \quad \text{for} \quad \tau \in (0, \tau_1^*), \tag{2.8b}$$

$$\frac{d}{d\tau}\left[e^{q\tau} P_1(\tau)\right] < 0 \quad \text{for} \quad \tau \in (\tau_1^*, \infty). \tag{2.8c}$$

The proof of Lemma 2.1 is given in the Appendix.

3. Analytic Price Formula of the Shout Floor

The pricing properties of the one-shout shout floor have been investigated by Cheuk and Vorst (1997). In this section, we go beyond their results by deriving the analytic price formula of the one-shout shout floor.

3.1 The price of the shout floor

Let $R_1(S, \tau)$ be the price function of the one-shout shout floor. Using the results in Lemma 2.1, the analytic representation of $R_1(S, \tau)$ can be derived. Since there is no strike price X appearing in the terminal payoff function and the obstacle function $SP_1(\tau)$ observes linear homogeneity in S, one can show that $R_1(S, \tau)$ is linearly homogeneous in S. We may write $R_1(S, \tau) = Sg(\tau), g(\tau)$ to be determined. Substituting the assumed form of $R_1(S, \tau)$ into Eq. (2.5), we obtain the following governing equation for $g(\tau)$.

$$\begin{aligned} &\frac{d}{d\tau}[e^{q\tau} g(\tau)] \geq 0, \quad g(\tau) \geq P_1(\tau), \\ &\frac{d}{d\tau}[e^{q\tau} g(\tau)][g(\tau) - P_1(\tau)] = 0, \\ &g(0) = 0. \end{aligned} \tag{3.1}$$

We solve for $g(\tau)$ under the following two separate cases:

(i) $r \leq q$.

By Eq. (2.7), $\frac{d}{d\tau}[e^{q\tau} P_1(\tau)]$ is strictly positive for all $\tau > 0$ and $P_1(0) = 0$; therefore, we can deduce that

$$g(\tau) = P_1(\tau), \quad \tau \in (0, \infty). \tag{3.2a}$$

(ii) $r > q$.

By Eqs. (2.8a,b), we deduce similarly that

$$g(\tau) = P_1(\tau) \quad \text{for} \quad \tau \in (0, \tau_1^*]. \tag{3.2b}$$

When $\tau > \tau_1^*$, we cannot have $g(\tau) = P_1(\tau)$ since this would lead to $\frac{d}{d\tau}[e^{q\tau}g(\tau)] = \frac{d}{d\tau}[e^{q\tau}P_1(\tau)] \geq 0$, a result contradicting to that of Eq. (2.8c). Hence, we must have $\frac{d}{d\tau}[e^{q\tau}g(\tau)] = 0$ for $\tau \in (\tau_1^*, \infty)$. Solving the differential equation and applying the auxiliary condition: $g(\tau_1^*) = P_1(\tau_1^*)$, we obtain

$$g(\tau) = e^{-q(\tau-\tau_1^*)}P_1(\tau_1^*) \quad \text{for} \quad \tau \in (\tau_1^*, \infty). \tag{3.2c}$$

The above results are summarized in Theorem 3.1.

Theorem 3.1 The price function of the one-shout shout floor $R_1(S,\tau)$ has the following analytic representation.

(i) If $r \leq q$ $\quad R_1(S,\tau) = SP_1(\tau), \qquad \tau \in (0,\infty)$.

(ii) If $r > q$,

$$R_1(S,\tau) = \begin{cases} SP_1(\tau), & \tau \in (0,\tau_1^*] \\ e^{-q(\tau-\tau_1^*)}SP_1(\tau_1^*), & \tau \in (\tau_1^*,\infty), \end{cases}$$

where τ_1^* is the unique solution to the following equation

$$\frac{d}{d\tau}[e^{q\tau}P_1(\tau)] = 0.$$

3.2 Optimal shouting policy of the shout floor

The optimal shouting policy of the one-shout shout floor depends on the sign of $\frac{d}{d\tau}[e^{q\tau}P_1(\tau)]$. When the sign is non-negative, we have $R_1(S,\tau) = SP_1(\tau)$, inferring that the holder should shout at once. This occurs either when (i) $r \leq q$, $\tau \in (0,\infty)$, or (ii) $r > q$, $\tau \leq \tau_1^*$. Conversely, when $r > q$ and $\tau > \tau_1^*$, Theorem 3.1 indicates that $R_1(S,\tau) > SP_1(\tau)$, so the holder should not shout under such scenario. In other words, when $r > q$, the holder should never shout when $\tau > \tau_1^*$ and shout at once when $\tau \leq \tau_1^*$.

4. Optimal Shouting Boundary for the Resettable Put Option

Unlike the shout floor, the analytic price formula for the resettable put option is not available. We examine the characterization of the optimal shouting boundary $S_1^*(\tau)$ of the one-shout resettable put option, with regard to the asymptotic behaviors at $\tau \to 0^+$ and $\tau \to \infty$, monotonicity behaviors, etc. It is most interesting to observe that the behaviors of $S_1^*(\tau)$ depend on the relative values of r and q. We also obtain the integral equation for the determination of $S_1^*(\tau)$ and solve the equation using the recursive integration method [Huang *et al.* (1996)].

4.1 Asymptotic behaviors of $S_1^*(\tau)$

For American options, it is well known that the critical asset price at $\tau \to 0^+$ depends on the ratio of r and q. However, this is not so for the resettable put option. Indeed, we have the following result:

Theorem 4.1 The optimal shouting boundary $S_1^*(\tau)$ for the one-shout resettable put option starts from X, namely, $S_1^*(0^+) = X$.

The mathematical proof of Theorem 4.1 is presented in the Appendix. From financial point of view, if $S_1^*(0^+) \neq X$, then arbitrage opportunities arise since the theta of the resettable put option would always be positive at $S = X$ as time is approaching expiry.

Next, we examine the asymptotic behaviors of the shouting boundary of the resettable put option $S_1^*(\tau)$ at infinite time to expiry. Let $S_{1,\infty}^*$ denote the limit of $S_1^*(\tau)$ as $\tau \to \infty$. We would like to show that $S_{1,\infty}^*$ exists when $r < q$, and subsequently determine its corresponding value. This is linked with the existence of the following limit

$$\lim_{\tau\to\infty} e^{r\tau} P_1(\tau) = \lim_{\tau\to\infty} [N(-d_2) - e^{(r-q)\tau} N(-d_1)] = 1 \quad \text{for} \quad r \leq q. \tag{4.1}$$

Let $V_1(S,\tau)$ be the price function of the one-shout resettable put option. We apply the transformation: $W_1(S,\tau) = e^{r\tau} V_1(S,\tau)$ to Eq. (2.6). In terms of $W_1(S,\tau)$, the transformed set of equations become

$$\begin{aligned}
&\frac{\partial W_1}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 W_1}{\partial S^2} - (r-q) S \frac{\partial W_1}{\partial S} = 0, \quad 0 < S < S_1^*, \quad \tau > 0, \\
&W_1(0,\tau) = X, \quad W_1(S_1^*,\tau) = S_1^* e^{r\tau} P_1(\tau), \\
&\frac{\partial W_1}{\partial S}(S_1^*,\tau) = e^{r\tau} P_1(\tau), \\
&W_1(S,\tau) = \max(X - S, 0).
\end{aligned} \tag{4.2}$$

Let $W_1^\infty(S)$ denote the limit of $W_1(S,\tau)$ as $\tau \to \infty$. The temporal derivative term $\dfrac{\partial W_1}{\partial \tau}$ in Eq. (4.2) vanishes upon taking the limit $\tau \to \infty$. The corresponding set of governing equations for $W_1^\infty(S)$ are given by

$$\begin{aligned}
&\frac{\sigma^2}{2} S^2 \frac{d^2 W_1^\infty}{dS^2} + (r-q) S \frac{dW_1^\infty}{dS} = 0, \quad 0 < S < S_{1,\infty}^*, \\
&W_1^\infty(0) = X, \quad W_1^\infty(S_{1,\infty}^*) = S_{1,\infty}^*, \\
&\frac{dW_1^\infty}{dS}(S_{1,\infty}^*) = 1.
\end{aligned} \tag{4.3}$$

This formulation for $W_1^\infty(S)$ implicitly requires the existence of $\lim\limits_{\tau\to\infty} e^{r\tau} P_1(\tau)$, so it is applicable only for $r \leq q$ [see Dewynne *et al.* (1989)]. The solution of Eq. (4.3) gives both $W_1^\infty(S)$ and $S_{1,\infty}^*$. The solution to $W_1^\infty(S)$ is found to be

$$W_1^\infty(S) = X + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} X^{-\alpha} S^{1+\alpha}, \quad 0 < S < S_{1,\infty}^*, \tag{4.4}$$

where $S_{1,\infty}^* = \left(1 + \dfrac{1}{\alpha}\right) X$ and $\alpha = 2(q-r)/\sigma^2$.

Hence, when $r < q$, $S_1^*(\tau)$ is defined for $\tau \in (0,\infty)$ with the asymptotic limit $S_{1,\infty}^* = \left(1 + \dfrac{1}{\alpha}\right) X$. Note that when $r = q$, α becomes zero; and correspondingly, $S_{1,\infty}^*$ becomes infinite.

When $r > q$, we recall the result in Theorem 3.1 that it is never optimal to shout the one-shout shout floor at $\tau > \tau_1^*$. Equivalently, $R_1(S,\tau) > SP_1(\tau)$ at $\tau > \tau_1^*$ when $r > q$. On the other hand, it is obvious that $V_1(S,\tau) \geq R_1(S,\tau)$ for all S and τ. Hence, when $r > q$, it is never optimal to shout at $\tau > \tau_1^*$ by virtue of the property $V_1(S,\tau) > SP_1(\tau)$ at $\tau > \tau_1^*$. In addition, due to Lemma 2.1 (ii), there always exists a critical asset price above which it is optimal to shout when $\tau < \tau_1^*$. More precisely, for $r > q$ and $\tau_0 < \tau_1^*$, one can show that there exists a critical asset price $S_1^*(\tau_0)$ such that $V_1(S,\tau_0) = SP_1(\tau_0)$ for $S \geq S_1^*(\tau_0)$. For the rigorous proof of the above statement, we can employ the comparison principle and other related techniques discussed in Brezis and Friedman (1976). Combination of above arguments yields the result that the optimal shouting boundary $S_1^*(\tau)$ is defined only for $\tau \in (0,\tau_1^*)$ if $r > q$.

We now summarize the above results as follows:

Theorem 4.2 The asymptotic behavior of the optimal shouting boundary $S_1^*(\tau)$ of the resettable put option at $\tau \to \infty$ depends on the relative values of r and q.

(i) If $r < q$, then $\lim_{\tau\to\infty} S_1^*(\tau) = \left[1 + \frac{\sigma^2}{2(q-r)}\right] X$.

(ii) If $r = q$, then $\lim_{\tau\to\infty} S_1^*(\tau) = \infty$.

(iii) If $r > q$, then $S_1^*(\tau)$ is defined only for $\tau \in (0,\tau_1^*)$, where τ_1^* is the unique solution of $\frac{d}{d\tau}[e^{q\tau}P_1(\tau)] = 0$.

4.2 Monotonicity of $S_1^*(\tau)$

In this subsection, we would like to establish the result that the critical asset price $S_1^*(\tau)$ is an increasing function of τ. We recall that similar monotonic property of the critical asset price is shared by the American put option. Since the intrinsic value $\max(X - S, 0)$ of the American put option is independent of time and longer-lived American put option is worth more than its shorter-lived counterpart, this leads directly to the monotonicity of the critical asset price of the American put option. However, the intrinsic value of the one-shout resettable put option is the price of the corresponding at-the-money put option and that price is time dependent. Therefore, it is not quite straightforward to observe the monotonicity of the optimal shouting boundary of the resettable put option.

To perform the analysis of the monotonicity of $S_1^*(\tau)$, we first introduce an auxiliary notion called the *spread value* of the resettable put option.

Definition 4.3 The *spread value* $D_1(S,\tau)$ of the resettable put option is defined as the price difference between the resettable put option and the corresponding at-the-money put option, namely,

$$D_1(S,\tau) = V_1(S,\tau) - SP_1(\tau).$$

We would like to choose the domain of definition of $D_1(S,\tau)$ so that cases where the holder never shouts her one-shout resettable put option are ruled out. Hence, the spread value $D_1(S,\tau)$ is defined for (i) $S > X$ and $\tau < \tau_1^*$ when $r > q$, and (ii) $S > X$ and $\tau < \infty$ when $r \leq q$. Generally speaking, the price function of the one-shout resettable put option has no monotonicity property with respect to time. However, $D_1(S,\tau)$ observes the properties as stated in Lemma 4.4.

Lemma 4.4 The spread value $D_1(S,\tau)$, with the domain of definition: (i) $S > X$ and $\tau < \tau_1^*$ when $r > q$, and (ii) $S > X$ and $\tau < \infty$ when $r \le q$, is monotonically increasing with τ and decreasing with S.

The proof of Lemma 4.4 is presented in the Appendix.

Suppose the one-shout resettable put option is shouted at $S_1^*(\tau)$ so that $D_1(S_1^*(\tau),\tau) = 0$, then from Lemma 4.4 one has $D_1(S,\tau) = 0$ for all $S \ge S_1^*(\tau)$. On the other hand, if one does not shout the one-shout resettable put option at S and τ', then $V(S,\tau') > SP_1(\tau')$ or $D_1(S,\tau') > 0$. From Lemma 4.4, we infer that $D_1(S,\tau) > 0$ for all $\tau > \tau'$, that is, one would never shout at the same value of S and at $\tau > \tau'$. Combining the above arguments, one obtains Theorem 4.5.

Theorem 4.5 The optimal shouting boundary $S_1^*(\tau)$ of the one-shout resettable put option is monotonically increasing with τ.

4.3 Integral equation for $S_1^*(\tau)$

We would like to find the integral representation of the shouting premium of the resettable put option. In the stopping region $S \ge S_1^*(\tau)$, we have $V_1(S,\tau) = SP_1(\tau)$. Hence, the governing equation for $V_1(S,\tau)$ for $S \in (0,\infty)$ is given by

$$\frac{\partial V_1}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_1}{\partial S^2} - (r-q)S\frac{\partial V_1}{\partial S} + rV_1 = \begin{cases} 0, & 0 < S < S_1^*(\tau), \\ Se^{-q\tau}\frac{d}{d\tau}[e^{q\tau}P_1(\tau)], & S \ge S_1^*(\tau). \end{cases} \tag{4.5}$$

Let $\psi(S_\xi;S)$ denote the transition density function for the future asset value S_ξ at ξ periods from now, given the current asset value S. The value of the resettable put option can be expressed as

$$\begin{aligned} V_1(S,\tau) &= e^{-r\tau}\int_0^\infty V_1(S_\tau,0)\psi(S_\tau;S)\,dS_\tau \\ &\quad + \int_0^\tau e^{-r\xi}\int_{S_1^*(\tau-\xi)}^\infty S_\xi e^{-q(\tau-\xi)}\frac{d}{du}\left[e^{qu}P_1(u)\right]\Big|_{u=\tau-\xi}\psi(S_\xi;S)\,dS_\xi d\xi, \end{aligned} \tag{4.6a}$$

where

$$\psi(S_\xi;S) = \frac{1}{S_\xi\sigma\sqrt{2\pi\xi}}\exp\left(-\frac{\left[\ln\frac{S_\xi}{S} - \left(r-q-\frac{\sigma^2}{2}\right)\xi\right]^2}{2\sigma^2\xi}\right). \tag{4.6b}$$

The first term is simply the value of the corresponding European put option $p_E(S,\tau)$, and the second term gives the shouting premium $E_1(S,\tau)$ of the resettable put option. The integral representation of $E_1(S,\tau)$ can be expressed as

$$E_1(S,\tau) = Se^{-q\tau}\int_0^\tau N(d_{1,\tau-u})\frac{d}{du}[e^{qu}P_1(u)]\,du, \quad u = \tau-\xi, \tag{4.7a}$$

where

$$d_{1,\tau-u} = \frac{\ln\frac{S}{S_1^*(u)} + \left(r-q+\frac{\sigma^2}{2}\right)(\tau-u)}{\sigma\sqrt{\tau-u}}. \tag{4.7b}$$

At the critical asset value $S = S_1^*(\tau), V_1(S,\tau) = SP_1(\tau)$. Substituting this relation into Eq. (4.6a), we obtain the following integral equation for $S_1^*(\tau)$:

$$S_1^*(\tau)P_1(\tau) = p_E(S_1^*(\tau),\tau) + S_1^*(\tau)e^{-q\tau}\int_0^\tau N(d_{1,\tau-u}^*)\frac{d}{du}[e^{qu}P_1(u)]\,du, \tag{4.8a}$$

where

$$d_{1,\tau-u}^* = \frac{\ln\frac{S_1^*(\tau)}{S_1^*(u)} + \left(r-q+\frac{\sigma^2}{2}\right)(\tau-u)}{\sigma\sqrt{\tau-u}}. \tag{4.8b}$$

One can then apply the recursive integration method (Huang *et al.*, 1996) to solve for $S_1^*(\tau)$ from the above integral equation. This is done by integrating the integral premium term using numerical quadrature and determining the optimal shouting boundary $S_1^*(\tau)$ at discrete instants t_k recursively. As a remark on the numerical implementation, since the integrand function inside the integral term has an integrable square root singularity at $u = 0$, it is necessary to transform the integral into the following form:

$$\begin{aligned}&\int_0^\tau N(d_{1,\tau-u}^*)\frac{d}{du}[e^{qu}P_1(u)]\,du\\ &= -(r-q)\int_0^\tau N(d_{1,\tau-u}^*)e^{-(r-q)u}N(-d_2)\,du\\ &\quad+\sigma\int_0^{\sqrt{\tau}} N(d_{1,\tau-u^2}^*)n(d_1(u^2))\,du.\end{aligned} \tag{4.9}$$

4.4 Numerical results

We applied both the binomial scheme (with the necessary dynamic programming procedure to incorporate the shouting feature) and the recursive integration method to determine the option value $V_1(S,\tau)$ and the optimal shouting boundary $S_1^*(\tau)$ of the one-shout resettable put option. In all calculations, we take the strike price $X = 1.0$ and volatility $\sigma = 20\%$.

FIGURES 1a and 1b show the plots of $V_1(S,\tau)$ against S at different values of τ, corresponding to $r < q$ and $r > q$, respectively. The price functions $V_1(S,\tau)$ show no monotonic property in τ. This behavior is in contrast to the American options where American option price functions are always monotonically increasing in τ. The lack of monotonicity in τ may be attributed to the fact that the derivative received upon shouting is an at-the-money European put option, and the price function of a European put option does not exhibit monotonicity in τ. For $r < q$, each price curve touches tangentially the line representing the value of the corresponding at-the-money put option (see FIGURE 1a). When $r > q$, there exists a critical value of τ above which it is never optimal to shout. When the following set of parameter values are used in the option model: $r = 0.06, q = 0.02, \sigma = 0.2$ and $X = 1$, this critical value of τ is found to be 5.7121. In FIGURE 1b, we observe that when $\tau < 5.7121$ (say, $\tau = 0.5$ or $\tau = 1.5$), the price curve touches the line representing the value of the at-the-money put. However, when $\tau > 5.7121$ (say, $\tau = 6.0$), the price curve always stays above the at-the-money put value line.

In FIGURES 2a, 2b and 2c, we plot the critical asset price $S_1^*(\tau)$ as a function of τ corresponding to $r < q, r = q$ and $r > q$, respectively. In all cases, $S_1^*(\tau)$ is a monotonically increasing function of τ. These plots verify the result stated in Theorem 4.5. Firstly, when $r < q$ (see FIGURE 2a), $S_1^*(\tau)$ is defined for $\tau \in (0, \infty)$ and $\lim_{\tau\to\infty} S_1^*(\tau) = 1.5$. This asymptotic value agrees with $S_1^*(\tau)$ as given in Theorem 4.2. Secondly, when $r = q$ (see FIGURE 2b), $S_1^*(\tau)$ tends to infinity as τ tends to infinity. Lastly, when $r > q$ (see FIGURE 2c), $S_1^*(\tau)$ is defined only for $\tau \in (0, \tau_1^*)$, where τ_1^* is obtained by solving Eq. (2.8a). Such behavior of $S_1^*(\tau)$ indicates that it is never optimal to shout when $\tau > \tau_1^*$.

5. Options with multiple shouting rights

In this section, we extend our discussion on the pricing models of resettable put options with multiple shouting rights throughout the life of the option contract. Let n denotes the total number of shouts allowed for the holder. Let t_j denote the time of the j^{th} shout to be chosen optimally by the holder of a n-shout resettable put option, and S_{t_j} denote the asset value at the shout instant t_j. Since the new reset strike price should be higher than the previous reset strike price, we should have $S_{t_j} > S_{t_{j-1}}$ and $S_{t_j} > X$ in all shouts. The terminal payoff of the n-shout resettable put option is given by $\max(S_{t_\ell} - S_T, 0)$, where t_ℓ is the last shouting instant chosen by the holder, $0 \le \ell \le n$. Similarly, the terminal payoff of the n-shout call option will be $\max(S_T - X, S_{t_\ell} - X)$.

One can apply similar argument as in Section 2.1 to deduce that the n-shout call option can be replicated by the combination of the n-shout resettable put option and a forward contract with the delivery price same as the strike price of the n-shout call option. Let $V_n(S,\tau)$ and $U_n(S,\tau)$ denote the price of the n-shout resettable put option and the n-shout call option, respectively. These two prices are related by

$$U_n(S,\tau) = V_n(S,\tau) + Se^{-q\tau} - Xe^{-r\tau}. \tag{5.1}$$

In the following discussion, we concentrate on the pricing model of the n-shout resettable put option.

5.1 Pricing formulation of the n-shout resettable put option

Upon shouting of the n-shout resettable put option, the number of reset rights is reduced by one so the resettable put option reduces to the one with only $n-1$ resets allowed. The resettable put option is at-the-money exactly at the instant of shouting. The price of an at-the-money resettable put option is linearly homogeneous in S. We define the function $P_n(\tau)$ to be

$$P_n(\tau) = V_{n-1}(1,\tau; X = 1), \tag{5.2}$$

so that the price of an at-the-money $(n-1)$-shout resettable put option is given by $SP_n(\tau)$.

The linear complementarity formulation of the free boundary value problem associated with the n-shout resettable put option can be expressed as [see also Eq. (2.5)]

$$\begin{aligned} &\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_n}{\partial S^2} - (r-q)S\frac{\partial V_n}{\partial S} + rV_n \ge 0, \quad V_n(S,\tau) \ge SP_n(\tau), \\ &\left[\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_n}{\partial S^2} - (r-q)S\frac{\partial V_n}{\partial S} + rV_n\right][V_n - SP_n(\tau)] = 0, \\ &V_n(S,0) = \max(X - S, 0). \end{aligned} \tag{5.3}$$

It is analytically intractable to perform the full theoretical analysis on the characterization of the optimal shouting policies for the multi-shout options. The main difficulty comes from the non-availability of the analytic forms for $P_n(\tau), n > 1$ [unlike $P_1(\tau)$ which has a simple analytic representation; see Eq. (2.3)]. However, the numerical calculations of the option values and optimal shouting boundaries of multi-shout options using the binomial method are quite straightforward. We first present our numerical calculations on the one-shout, two-shout and three-shout resettable put options under the two cases: $r < q$ and $r > q$.

5.2 Properties of the price functions and optimal shouting boundaries of n-shout resettable put options from numerical experiments

Since a n-shout resettable put option is reduced to its $(n-1)$-shout counterpart upon shouting, one may expect that the characterization of the shouting policies of the n-shout resettable put option should resemble closely to that of the one-shout counterpart. Let $S_n^*(\tau)$ denote the critical asset price for the n-shout resettable put option. We first postulate some properties on $S_n^*(\tau)$, and then examine their validity through numerical experiments.

When $r < q$, the optimal shouting boundary exists at all times, that is, $S_n^*(\tau)$ is defined for $\tau \in (0, \infty)$. For a given value of τ, one observes $S_{n+1}^*(\tau) < S_n^*(\tau), n = 1, 2, \cdots$. This can be deduced from the financial intuition that the holder should choose to shout at a higher critical asset price with less allowable shouts remaining. The shouting boundaries start at X, that is, $S_n^*(0^+) = X$, and $S_n^*(\tau)$ is an increasing function of τ with a finite asymptotic value at $\tau \to \infty$. Further, from the monotonic property on n, we have

$$\lim_{\tau \to \infty} S_{n+1}^*(\tau) < \lim_{\tau \to \infty} S_n^*(\tau), \quad n = 1, 2, \cdots. \tag{5.4}$$

In the special case $r = q, S_n^*(\tau) \to \infty$ as $\tau \to \infty, n = 1, 2, \cdots$.

When $r > q$, $S_n^*(\tau)$ retains the monotonic properties in both n and τ and $S_n^*(\tau)$ also starts at X. However, $S_n^*(\tau)$ is defined only for $\tau \in [0, \tau_n^*)$, where τ_n^* is the critical value for τ such that it is never optimal for the holder to shout the n-shout resettable put whenever $\tau > \tau_n^*$. With less number of shouts remaining, the holder would become more conservative on the use of the shouting rights. For a given τ, it may occur that it would be optimal to shout a n-shout resettable put at sufficiently high asset price level but not so for its $(n-1)$-shout counterpart. Hence, we expect $\tau_{n+1}^* > \tau_n^*, n = 1, 2, \cdots$.

We applied the binomial method, together with the dynamic programming procedure for the shout feature, to compute the option values and critical asset prices of one-shout, two-shout and three-shout resettable put options. The strike price and volatility are taken to be $X = 1.0$ and $\sigma = 0.2$ in all calculations. For $r < q$, we take $r = 0.02$ and $q = 0.06$; while for $r > q$, we take $r = 0.06$ and $q = 0.02$.

In FIGURE 3a, we plot $V_1(S,\tau), V_2(S,\tau)$ and $V_3(S,\tau)$ against S at $\tau = 1$, given $r < q$. We observe the monotonic property $V_1(S,\tau) < V_2(S,\tau) < V_3(S,\tau)$, which agrees with the intuition that put option with more shouting rights should have higher values. At the critical asset prices, all of the price curves touch tangentially the line representing the value of the corresponding at-the-money put option. The price function of the at-the-money $(n-1)$-shout put option is given by $SP_n(\tau)$ [see Eq. (5.2)]. The critical asset prices, $S_1^*(\tau), S_2^*(\tau)$ and $S_3^*(\tau)$, corresponding to the one-shout, two-shout and three-shout put options, observe the monotonic property: $S_1^*(\tau) > S_2^*(\tau) > S_3^*(\tau)$.

In FIGURE 3b, we plot $V_1(S,\tau), V_2(S,\tau)$ and $V_3(S,\tau)$ against S at $\tau = 12$, given $r > q$. All of the monotonic properties stated in the last paragraph remain valid even when $r > q$. At $\tau = 12$, only the price curve $V_3(S,\tau)$ touches the corresponding at-the-money put option value line. The price curves $V_1(S,\tau)$ and $V_2(S,\tau)$ always stay above the corresponding at-the-money put option value lines, implying that it is never optimal to shout at any asset price level. When $r > q$, there exists a critical value τ_n^* for the n-shout resettable put option such that the holder never shouts when $\tau > \tau_n^*$. One then infers from the properties of these three price curves that $\tau = 12$ lies between τ_2^* and τ_3^*, that is, $\tau_1^* < \tau_2^* < 12 < \tau_3^*$.

FIGURES 4a and 4b show the plots of $S_n^*(\tau)$ against $\tau, n = 1, 2, 3$ for $r < q$ and $r > q$, respectively. When $r < q$, we observe that $S_n^*(\tau)$ is defined for $\tau \in (0, \infty)$ and $S_{n+1}^*(\tau) < S_n^*(\tau), n = 1, 2$. Also, $S_n^*(\tau)$ tends to a finite asymptotic value as $\tau \to \infty, n = 1, 2, 3$. From FIGURE 4a, these asymptotic values are approximately found to be $1.5, 1.31$ and 1.23, for the one-shout, two-shout and three-shout resettable put options, respectively. On the other hand, when $r > q$, the shouting boundaries in FIGURE 4b reveal that $S_n^*(\tau)$ is defined only for $\tau \in (0, \tau_n^*), n = 1, 2, 3$. These critical values are estimated to be $\tau_1^* \approx 5.71, \tau_2^* \approx 9.55$ and $\tau_3^* \approx 13.0$ for the one-shout, two-shout and three-shout resettable put options, respectively.

5.3 Properties of $P_n(\tau)$

The functions $P_n(\tau), n = 1, 2, \cdots$ [see the definition in Eq. (5.2)] play an important role in determining the optimal shouting policies of the n-shout shout floors and the n-shout resettable put options. The properties of $P_n(\tau)$ are summarized in Lemma 5.1.

Lemma 5.1 The function $P_n(\tau)$ observes the following properties.

(i) If $r \leq q$, then

$$\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \infty). \tag{5.5}$$

(ii) If $r > q$, there exists a unique critical value $\tau_n^* \in (0, \infty)$ such that

$$\left.\frac{d}{d\tau}[e^{q\tau}P_n(\tau)]\right|_{\tau=\tau_n^*} = 0, \tag{5.6a}$$

and

$$\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \tau_n^*), \tag{5.6b}$$

$$\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] < 0 \quad \text{for} \quad \tau \in (\tau_n^*, \infty). \tag{5.6c}$$

In addition, we have $\tau_n^* < \tau_{n+1}^*$ and $\lim_{n\to\infty} \tau_n^* = \infty$.

Due to the lack of the analytic formula of $P_n(\tau)$, the proof is not given here since this requires theoretical tools in partial differential equation theory to establish the result. However, their validity has been verified by numerical results (see FIGURES 3a, 3b, 4a and 4b).

5.4 Price functions and optimal shouting policies of the n-shout shout floor

As a consequence of Lemma 5.1, one may deduce the price function of the n-shout shout floor. Upon the first shouting, the n-shout shout floor then becomes the corresponding at-the-money

$(n-1)$-shout resettable put option. The n-shout shout floor shares the same governing equation with that of n-shout resettable put option [see Eq. (5.3)] except that the initial condition is set to be zero. Following similar argument as that for the one-shout case (see Sec. 3), the price function takes different forms according to $r \leq q$ or $r > q$ (see Theorem 5.2).

Theorem 5.2 Let $R_n(S,\tau)$ denote the price function of the n-shout shout floor. We have

(i) If $r \leq q$, $\quad R_n(S,\tau) = SP_n(\tau), \quad \tau \in (0,\infty).$

(ii) If $r > q$,

$$R_n(S,\tau) = \begin{cases} SP_n(\tau) & \tau \in (0,\tau_n^*] \\ e^{-q(\tau-\tau_n^*)}SP_n(\tau_n^*), & \tau \in (\tau_n^*,\infty), \end{cases}$$

where τ_n^* is the unique solution to $\dfrac{d}{d\tau}[e^{q\tau}P_n(\tau)] = 0$.

Remark The above analytic representation of $R_n(S,\tau)$ is not an analytic formula in a strict sense. Recall that $P_n(\tau) = V_{n-1}(1,\tau;X=1)$, so one has to find $V_1(S,\tau), V_2(S,\tau), \cdots$, successively in order to obtain $P_n(\tau)$.

Optimal shouting policies

First, consider the case $r \leq q$. Since we have $R_n(S,\tau) = SP_n(\tau)$ for all values of τ, we deduce that the first shouting right will be utilized at once at any time and any asset price level. Next, when $r > q$, the n-shout shout floor will not be shouted at any asset price when $\tau > \tau_n^*$. However, it will be shouted at once at any asset price level once $\tau \leq \tau_n^*$. Once the first shouting has occurred, the n-shout shout floor reduces to the at-the-money $(n-1)$-shout resettable put option. The subsequent optimal shouting policies will be governed by those of the multi-shout resettable put option (see Sec. 5.5).

5.5 Optimal shouting policies of the n-shout resettable put option

Similar to Definition 4.3, we define the spread value $D_n(S,\tau)$, $n \geq 2$ of the n-shout resettable put option as follows:

$$D_n(S,\tau) = V_n(S,\tau) - SP_n(\tau).$$

for (i) $S > X$ and $\tau < \tau_n^*$ when $r > q$, and (ii) $S > X$ and $\tau < \infty$ when $r \leq q$.

Similar to Lemma 4.4, one can show

Lemma 5.3 The spread value $D_n(S,\tau)$, with the domain of definition: (i) $S > X$ and $\tau < \tau_n^*$ when $r > q$, and (ii) $S > X$ and $\tau < \infty$ when $r \leq q$, is monotonically increasing with τ and decreasing with S.

The properties of the optimal shouting boundary $S_n^*(\tau)$ of the n-shout resettable put option are similar to those of the one-shout resettable put option. They are summarized in Theorem 5.4.

Theorem 5.4 Let $S_n^*(\tau)$ be the optimal shouting boundary of the n-shout resettable put option and let $S_{n,\infty}^*$ denote $\lim_{\tau\to\infty} S_n^*(\tau)$. The properties of $S_n^*(\tau)$ are given by

(i) $S_n^*(0^+) = X$.

(ii) If $r < q$, then

(5.7a) $$S^*_{n,\infty} = \left(1 + \frac{1}{\alpha}\right)\frac{X}{\beta_n}$$

where $\alpha = 2(q-r)/\sigma^2$, $\beta_1 = 1$ and

(5.7b) $$\beta_n = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta^{1+\alpha}_{n-1}.$$

In addition, we have the monotonic property $S^*_{n,\infty} < S^*_{n-1,\infty}, n = 2, 3, \cdots$, and $\lim_{n\to\infty} S^*_{n,\infty} = X$.

(iii) If $r = q$, then $S^*_{n,\infty}$ becomes infinite.

(iv) If $r > q$, then there exists a critical value $\tau^*_n \in (0, \infty)$ such that $S^*_n(\tau)$ is only defined for $\tau \in (0, \tau^*_n)$.

(v) $S^*_n(\tau)$ is monotonically increasing with τ.

(vi) $S^*_n(\tau) \leq S^*_{n-1}(\tau)$ for all $n \geq 2$.

The proof of part (i) is similar to that of Theorem 4.1. Using the results in part (ii), Theorem 5.2, and applying similar argument used in the one-shout model (see Sec. 4.1), we can establish part (iv). Part (iii) follows immediately from part (ii) by observing that $\alpha = 0$ when $r = q$. Similar to Theorem 4.5, the proof of part (v) can be derived from Lemma 5.3. The proofs of part (ii) and (vi) are presented in the Appendix.

6. Conclusion

The shout feature embedded in a derivative entitles the holder the right to reset certain terms in the derivative contract. This may be interpreted as the privilege given to the holder to convert the original derivative to a new derivative or asset, and the time to shout is chosen optimally by the holder. In a broad sense, the common early exercise feature in American options and conversion feature in convertible bonds can be visualized as special forms of the shouting feature. Since the critical asset price at which the holder optimally shouts is not known a prior but has to be determined in the solution process, the pricing models are formulated as free boundary value problems.

In this paper, we have analyzed the optimal shouting policies of options with single or multiple shouting rights. The resemblances between the shout call option and the resettable put option have been examined. The behaviors of the optimal shouting boundaries of the resettable put options depend crucially on the relative values of the riskless interest rate r and dividend yield q. When $r \leq q$, the shouting boundary is defined at all times. This implies that at any time during the life of the option, the holder should choose to shout optimally when the asset value goes above some threshold value. On the other hand, when $r > q$, there exist a critical time earlier than which it is never optimal for the holder to shout the resettable put option at any asset value level. The optimal shouting policies of the multi-shout shout floor have striking properties. When $r \leq q$, the shout floor should be shouted at once at any time and at any asset price level. When $r > q$, there exists a critical time earlier than which it is never optimal for the holder to shout the shout floor. However, the shout floor should be shouted at once at any asset price upon reaching the critical time. When the first shouting has occurred in a multi-shout shout floor, the shout floor becomes the corresponding at-the-money resettable put option with one shouting right less.

A number of interesting analytic formulas have been derived in the paper. We obtain the closed form price formula of the one-shout shout floor and the integral representation of the shouting premium of the one-shout resettable put option. The analytic representation of the price of the multi-shout shout floor is also deduced. In addition, we obtain the asymptotic critical asset prices at infinite time to expiry for the one- and multi-shout resettable put options.

Several results on the monotonic properties with regard to the critical asset prices and shouting boundaries have been explored. All these monotonic properties agree with the corresponding financial intuitions. Some of these properties are: (1) an option with more number of shouting rights should have higher value compared to its counterpart with less; (2) the holder shouts at a lower critical asset price with more shouting rights outstanding; (3) the holder chooses to shout at a lower critical asset price for a shorter-lived option; (4) the critical time earlier than which it is never optimal to shout increases with more shouting rights outstanding. These results are established through verification by numerical experiments and theoretical justification by rigorous mathematical proofs.

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APPENDIX

Proof of Lemma 2.1 The derivative of $e^{q\tau}P_1(\tau)$ is found to be

$$\frac{d}{d\tau}[e^{q\tau}P_1(\tau)] = e^{-(r-q)\tau}\left[-(r-q)N(-d_2) + \frac{\sigma}{2\sqrt{\tau}}n(d_2)\right],$$

where $n(x) = N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Obviously, when $r \leq q$, we have the property as stated in Eq. (2.7), that is,

$$\frac{d}{d\tau}[e^{q\tau}P_1(\tau)] > 0 \qquad \text{for} \quad \tau \in (0, \infty).$$

Next, we consider the case where $r > q$. It is necessary to consider the following three separate cases: (i) $r - q = \frac{\sigma^2}{2}$, (ii) $r - q > \frac{\sigma^2}{2}$ and (iii) $r - q < \frac{\sigma^2}{2}$.

First, for the special case $r - q = \frac{\sigma^2}{2}$, the derivative of $e^{q\tau}P_1(\tau)$ becomes $\frac{\sigma}{2}e^{-\sigma^2\tau/2}\left(\frac{1}{\sqrt{\tau}} - \frac{\sigma}{2}\right)$. The conditions stated in Eqs. (2.8a,b,c) are easily seen to be satisfied. In this case, $\tau_1^* = 4/\sigma^2$.

For the general case $r - q \neq \frac{\sigma^2}{2}$, we need to consider the property of the function

$$f(\tau) = -(r-q)N(-d_2) + \frac{\sigma}{2\sqrt{\tau}}n(d_2).$$

The derivative of $f(\tau)$ is found to be

$$f'(\tau) = \frac{r - q - \frac{\sigma^2}{2}}{4\sigma\sqrt{\tau}}\left[\left(r - q + \frac{\sigma^2}{2}\right) - \frac{\sigma^2}{\left(r - q - \frac{\sigma^2}{2}\right)\tau}\right]n(d_2).$$

It is seen that $f(\tau) \to \infty$ as $\tau \to 0^+$ and $f(\tau)$ always remains negative when τ becomes larger than some threshold value; so $f(\tau)$ must have at least one root in $(0, \infty)$. To show the validity of conditions stated in Eqs. (2.8a,b,c), it suffices to show that $f(\tau)$ has exactly one root. When $r - q > \frac{\sigma^2}{2}$, $f'(\tau) = 0$ has the *unique* solution $\widehat{\tau} = \frac{\sigma^2}{(r-q)^2 - \frac{\sigma^4}{4}}$. The function $f(\tau)$ is seen to have its absolute minima at $\tau = \widehat{\tau}$ since $f'(\tau) < 0$ for $\tau \in (0, \widehat{\tau})$ and $f'(\tau) > 0$ for $\tau \in (\widehat{\tau}, \infty)$. Together with the properties that $f(0^+) \to \infty$ and $\lim_{\tau\to\infty} f(\tau) < 0$, we conclude that $f(\tau)$ has exactly one root in $(0, \infty)$. For the remaining case $r - q < \frac{\sigma^2}{2}$, it is observed that $f(\tau)$ is a monotonic decreasing function of τ since $f'(\tau) < 0$ for $\tau \in (0, \infty)$. Hence, the same conclusion that exactly one root in $(0, \infty)$ is obtained.

Proof of Theorem 4.1 Let $D_1(S, \tau)$ denote the difference between the values of the resettable put option and its corresponding at-the-money put option, that is, $D_1(S, \tau) = V_1(S, \tau) - SP_1(\tau)$.

Substituting the above relation into Eq. (2.6), we obtain

$$\begin{aligned}
&\frac{\partial D_1}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D_1}{\partial S^2} - (r-q)S\frac{\partial D_1}{\partial S} + rD_1 = -S[P_1'(\tau) + qP_1(\tau)], \quad 0 < S < S_1^*(\tau), \tau > 0,\\
&D_1(0,\tau) = Xe^{-r\tau}, D_1(S_1^*(\tau),\tau) = 0,\\
&\frac{\partial D_1}{\partial S}(S_1^*(\tau),\tau) = 0,\\
&D_1(S,0) = \max(X-S,0).
\end{aligned}$$

Note that $D_1(S,\tau) \geq 0$ for all asset values and at all times. On the other hand, we observe that $-S[P_1'(\tau) + qP_1(\tau)] \to -\infty$ as $\tau \to 0^+$. Assuming $S_1^*(0^+) > X$, then for $S \in (X, S_1^*(0^+))$ we have

$$\frac{\partial D_1}{\partial \tau}(S, 0^+) = -S[P_1'(0^+) + qP_1(0)] < 0.$$

This would imply $D_1(S,0^+) < 0$, a contraction to $D_1(S,\tau) \geq 0$ at all times. Therefore, we must have $S_1^*(0^+) \leq X$. On the other hand, financial intuition dictates that $S_1^*(0^+) \geq X$ (see Sec. 4.1). Combining the results, we obtain $S_1^*(0^+) = X$.

Proof of Lemma 4.4 Let $V_1(S,\tau)$ and $p_E(S,\tau)$ be the price function of the one-shout resettable put option and the corresponding European put option, respectively. It is clear that $V_1(S,\tau) \geq p_E(S,\tau) \geq SP_1(\tau)$ for $S \leq X$. Hence, for $S \leq X$, the constraint $V_1(S,\tau) \geq SP_1(\tau)$ in the linear complementarity formulation for $V_1(S,\tau)$ can be replaced by $V_1(S,\tau) \geq p_E(S,\tau)$. For convenience of analysis, we define the continuation of the spread value $D_1(S,\tau)$ as follows:

$$D_1(S,\tau) = \begin{cases} V_1(S,\tau) - SP_1(\tau), & S > X \\ V_1(S,\tau) - p_E(S,\tau), & S \leq X, \end{cases}$$

which satisfies

$$\begin{aligned}
&\frac{\partial D_1}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D_1}{\partial S^2} - (r-q)S\frac{\partial D_1}{\partial S} + rD_1 \geq h_1(S,\tau), \quad D_1(S,\tau) \geq 0,\\
&\left[\frac{\partial D_1}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D_1}{\partial S^2} - (r-q)S\frac{\partial D_1}{\partial S} + rD_1 - h_1(S,\tau)\right] D_1(S,\tau) = 0,\\
&D_1(S,0) = 0,
\end{aligned}$$

where

$$h_1(S,\tau) = \begin{cases} -Se^{-q\tau}\dfrac{d}{d\tau}[e^{q\tau}P_1(\tau)], & S > X \\ 0, & S \leq X. \end{cases}$$

Due to Lemma 2.1, we have $\dfrac{\partial h_1}{\partial S} \leq 0$ for $\tau \in (0,\infty)$ if $r \leq q$ and for $\tau \in (0, \tau_1^*)$ if $r > q$. We then use the comparison principle of variational inequality to infer that $D_1(S,\tau)$ is monotonically decreasing with S. Next, we show that $D_1(S,\tau)$ is monotonically increasing with τ.

We first consider the case $r > q$ and follow the notations in Lemma 2.1. The proof of Lemma 2.1 implies that

$$\frac{d}{d\tau}(e^{-q\tau}\frac{d}{d\tau}[e^{q\tau}P_1(\tau)]) = -re^{-r\tau}f'(\tau) + e^{-r\tau}f''(\tau) < 0$$

for $\tau \in (0, \tau_1^*)$, so $\dfrac{\partial h_1}{\partial \tau} \geq 0$ for $\tau \in (0, \tau_1^*)$. The monotonicity of $D_1(S,\tau)$ with respect to τ is then established.

If $r \leq q$, we make the transformation $\widehat{D}_1(S,\tau) = e^{-\gamma\tau} D_1(S,\tau)$ in the above linear complementarity formulation and obtain

$$\frac{\partial \widehat{D}_1}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 \widehat{D}_1}{\partial S^2} - (r-q) S \frac{\partial \widehat{D}_1}{\partial S} + (r+\gamma)\widehat{D}_1 \geq \widehat{h}_1(S,\tau), \quad \widehat{D}_1(S,\tau) \geq 0,$$

$$\left[\frac{\partial \widehat{D}_1}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 \widehat{D}_1}{\partial S^2} - (r-q) S \frac{\partial \widehat{D}_1}{\partial S} + (r+\gamma)\widehat{D}_1 - \widehat{h}_1(S,\tau)\right] \widehat{D}_1(S,\tau) = 0,$$

$$\widehat{D}_1(S,0) = 0,$$

where γ is a positive constant and

$$\widehat{h}_1(S,\tau) = \begin{cases} -Se^{-(\gamma+q)\tau} \dfrac{d}{d\tau}[e^{q\tau} P_1(\tau)], & S > X \\ 0, & S \leq X. \end{cases}$$

It is not hard to check that we can choose $\gamma > 0$ to be sufficiently large to guarantee that,

$$\frac{d}{d\tau}\left(e^{-(\gamma+q)\tau} \frac{d}{d\tau}[e^{q\tau} P_1(\tau)]\right) = e^{-(r+\gamma)\tau}[-(r+\gamma)f + f']$$

$$= e^{-(r+\gamma)\tau}\left[(r+\gamma)(r-q)N(-d_2) + \frac{n(d_2)}{4\sigma\sqrt{\tau}}\left[-2\sigma^2(r+\gamma) - \frac{\sigma^4}{4} - \frac{\sigma^2}{\tau} + (r-q)^2\right]\right] \leq 0,$$

so that $\dfrac{\partial \widehat{h}_1}{\partial \tau} \geq 0$. Again, we use the comparison principle to give the result that $\widehat{D}_1(S,\tau)$ is monotonically increasing with respect to τ, so does $D_1(S,\tau)$. This completes the proof.

Proof of Theorem 5.4, part (ii) We follow similar approach as used for the one-shout model (see Sec. 4.1). Let $W_n(S,\tau) = e^{r\tau} V_n(S,\tau)$, and $W_n^\infty(S)$ denote the limit of $W_n(S,\tau)$ as $\tau \to \infty$. The governing equation for $W_n^\infty(S)$ takes the form

$$\frac{\sigma^2}{2} S^2 \frac{d^2 W_n^\infty}{dS^2} + (r-q) S \frac{dW_n^\infty}{dS} = 0, \qquad 0 < S < S_{n,\infty}^*,$$

$$W_n^\infty(S_{n,\infty}^*) = \beta_n S_{n,\infty}^*,$$

$$\frac{dW_n^\infty}{dS}(S_{n,\infty}^*) = \beta_n,$$

where $\beta_n = \lim_{\tau\to\infty} e^{r\tau} P_n(\tau)$ and $S_{n,\infty}^* = \lim_{\tau\to\infty} S_n^*(\tau)$. It has been seen that β_1 exists [see Eq. (3.1)]. In general, we have

$$\beta_n = \lim_{\tau\to\infty} e^{r\tau} P_n(\tau) = \lim_{\tau\to\infty} W_{n-1}(1,\tau;1) = W_{n-1}^\infty(1;1).$$

Hence, β_n exists provided that $W_{n-1}^\infty(1;1)$ is defined. The existence of β_n can be argued as follows. Given the existence of β_1, we can determine $W_1^\infty(1;1)$. This guarantees the existence of β_2, and from which we can determine $W_2^\infty(1;1)$, and so forth.

The general solution $W_n^\infty(S)$ is found to be

$$W_n^\infty(S) = X + CS^{1+\alpha},$$

where $\alpha = 2(q-r)/\sigma^2$ and C is an arbitrary constant. Applying the two auxiliary conditions, we obtain

$$C = \frac{1}{1+\alpha}\frac{\beta_n}{S_{n,\infty}^{*\alpha}} = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{\beta_n^{1+\alpha}}{X^\alpha},$$
$$S_{n,\infty}^* = \left(1+\frac{1}{\alpha}\right)\frac{X}{\beta_n}.$$

The recurrence relation for β_n is deduced to be

$$\beta_n = W_{n-1}^\infty(1;1) = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta_{n-1}^{1+\alpha}.$$

The monotonic relation $\beta_n > \beta_{n-1}$ leads to the monotonic property $S_{n-1,\infty}^* > S_{n,\infty}^*$. Taking the limit $n \to \infty$ in the above recurrence relation for β_n gives $\lim_{n\to\infty} \beta_n = 1 + 1/\alpha$. Correspondingly, this implies $\lim_{n\to\infty} S_{n,\infty}^* = X$. The first few values of β_n and $S_{n,\infty}^*$ are listed below:

(i) when $n = 1, \beta_1 = 1$ and $S_{1,\infty}^* = \left(1+\frac{1}{\alpha}\right)X$;

(ii) when $n = 2, \beta_2 = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$ and $S_{2,\infty}^* = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}}$;

(iii) when $n = 3$, $\beta_3 = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\left[1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\right]$ and $S_{3,\infty}^* = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\left[1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\right]}$.

Proof of Theorem 5.4, part (vi) Let $D_n(S,\tau)$ be the spread value of the n-shout resettable put option. It suffices to show that $D_n(S,\tau) \le D_{n-1}(S,\tau)$. Define the continuation of $D_n(S,\tau)$ as follows

$$D_n(S,\tau) = \begin{cases} V_n(S,\tau) - SP_n(\tau), & S > X \\ V_n(S,\tau) - V_{n-1}(S,\tau), & S \le X. \end{cases}$$

Note that $D_n(S,\tau)$ satisfies

$$\frac{\partial D_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D_n}{\partial S^2} - (r-q)S\frac{\partial D_n}{\partial S} + rD_n \ge h_n(S,\tau), \quad D_n(S,\tau) \ge 0,$$
$$\left[\frac{\partial D_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D_n}{\partial S^2} - (r-q)S\frac{\partial D_n}{\partial S} + rD_n - h_n(S,\tau)\right] D_n(S,\tau) = 0,$$
$$D_n(S,0) = 0,$$

where

$$h_n(S,\tau) = \begin{cases} -Se^{-q\tau}\frac{d}{d\tau}[e^{q\tau}P_n(\tau)], & S > X \\ 0, & S \le X \end{cases}.$$

Due to Lemma 5.3, we have $\frac{d}{d\tau}P_n(\tau) \ge \frac{d}{d\tau}P_{n-1}(\tau)$, so $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] \ge \frac{d}{d\tau}[e^{q\tau}P_{n-1}(\tau)]$, implying $h_n(S,\tau) \le h_{n-1}(S,\tau)$. The desired result is then obtained immediately by the comparison principle.

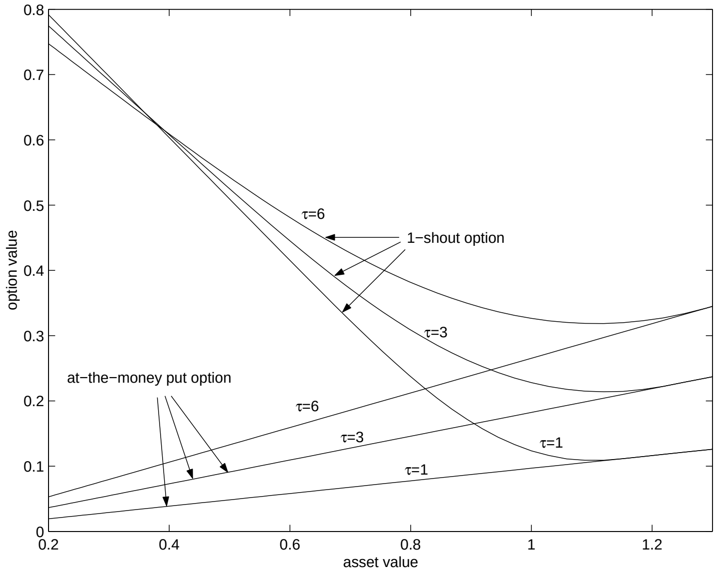

FIGURE 1a. Plot of the value of the one-shout resettable put option against the asset value for $r < q$ at different values of time to expiry, τ. The parameter values used in the calculations are: $r = 0.02, q = 0.06, \sigma = 0.2$ and $X = 1.0$. Each price curve touches tangentially the line representing the value of the corresponding at-the-money put option.

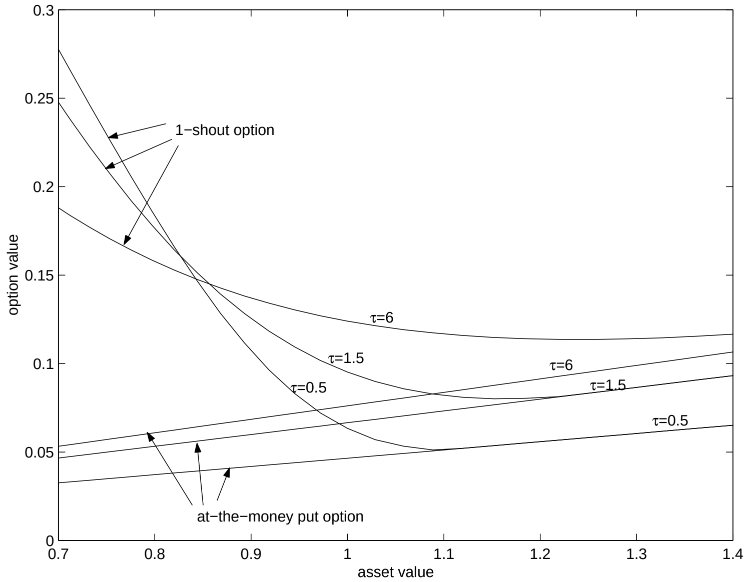

FIGURE 1b. Plot of the value of the one-shout resettable put option against the asset value for $r > q$ at different values of time to expiry, τ. The parameter values used in the calculations are: $r = 0.06, q = 0.02, \sigma = 0.2$ and $X = 1.0$. The critical value of time to expiry beyond which it is never optimal to shout is found to be 5.7121. The price curve corresponding to $\tau = 6$ (which is greater than 5.7121) never touches the line representing the value of the at-the-money put option.

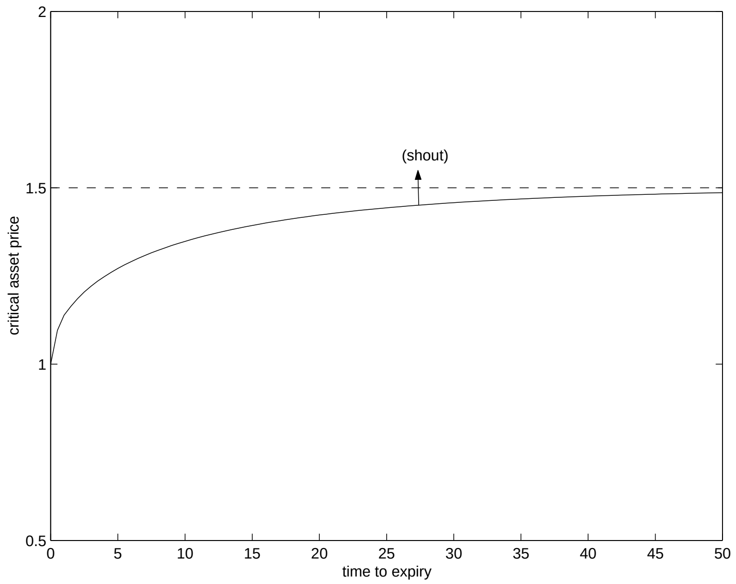

FIGURE 2a. Plot of the shouting boundary of the one-shout resettable put option as a function of time to expiry for $r < q$. The parameter values used in the calculations are: $r = 0.02, q = 0.06, \sigma = 0.2$ and $X = 1.0$. The asymptotic value of the critical asset price at infinite time to expiry is found to be 1.5.

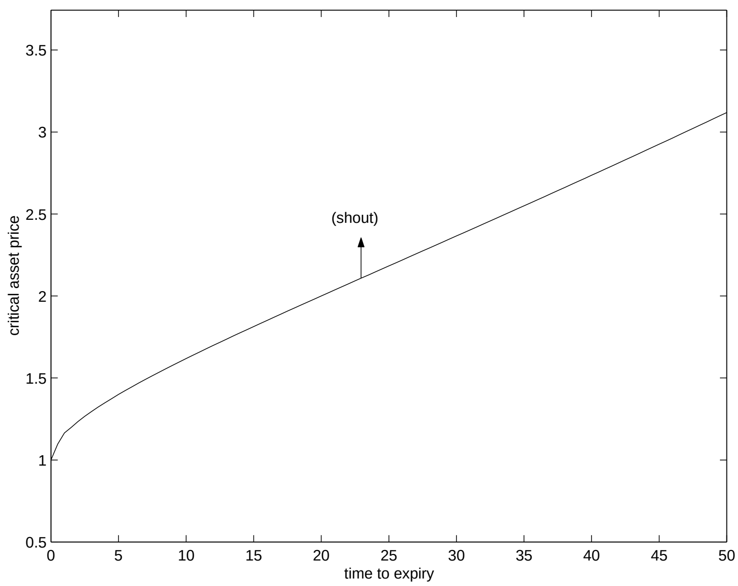

FIGURE 2b. Plot of the shouting boundary of the one-shout resettable put option as a function of time to expiry for $r = q$. The parameter values used in the calculations are: $r = 0.06, q = 0.06, \sigma = 0.2$ and $X = 1.0$. The critical asset price increases monotonically with increasing time to expiry and tends to infinity at infinite time to expiry.

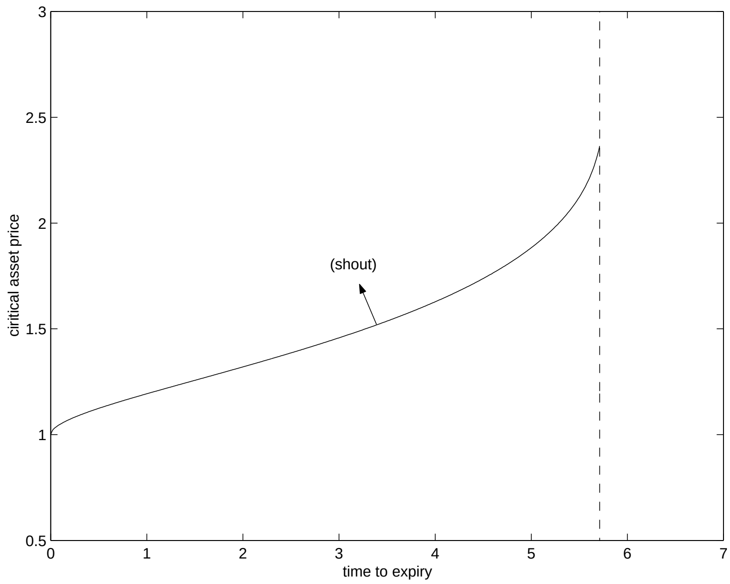

FIGURE 2c. Plot of the shouting boundary of the one-shout resettable put option as a function of time to expiry for $r > q$. The parameter values used in the calculations are: $r = 0.06, q = 0.02, \sigma = 0.2$ and $X = 1.0$. The critical value of the time to expiry beyond which it is never optimal to shout is found to be 5.7121.

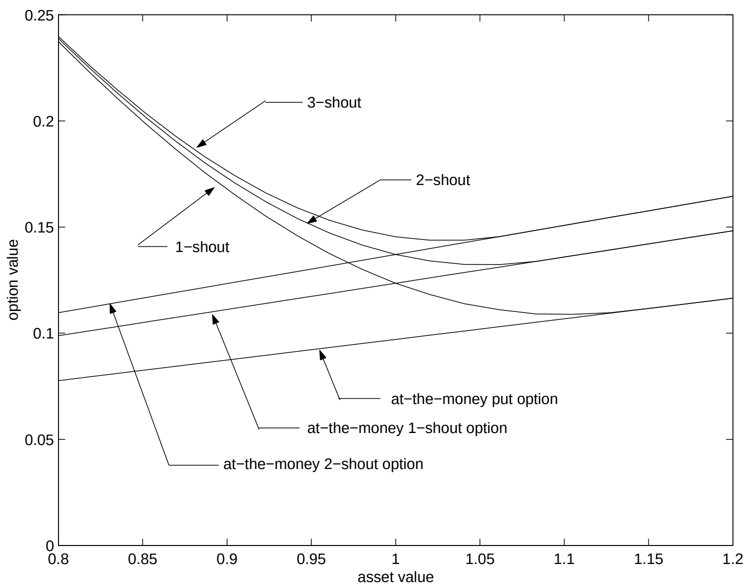

FIGURE 3a. Plot of the value of the resettable put options with the right of one, two and three shouts against the asset value at $\tau = 1$, given $r < q$. The parameter values used in the calculations are: $r = 0.02, q = 0.06, \sigma = 0.2$ and $X = 1.0$. Each price curve touches tangentially the line representing the value of the corresponding at-the-money put option.

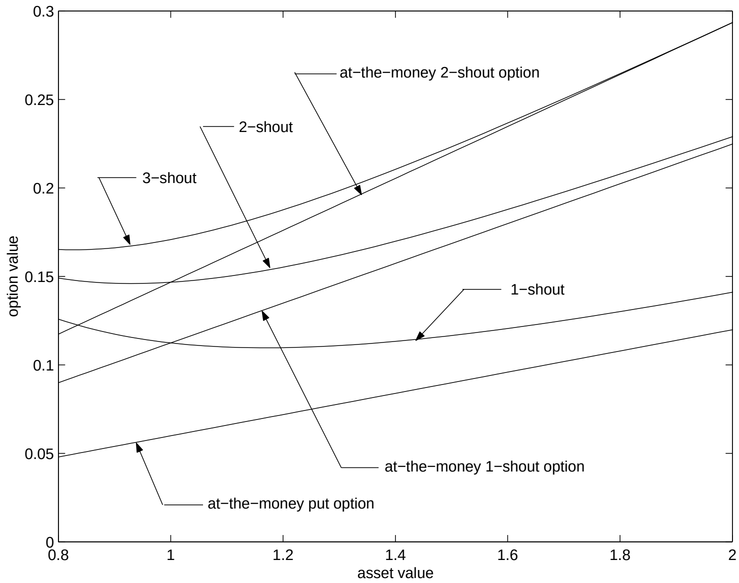

FIGURE 3b. Plot of the value of the resettable put options with the right of one, two and three shouts against the asset value at $\tau = 12$, given $r > q$. The parameter values used in the calculations are: $r = 0.06, q = 0.02, \sigma = 0.2$ and $X = 1.0$. The price curve corresponding to the three-shout option touches the value line of the corresponding at-the-money put option, while the price curves corresponding to the one-shout and two-shout options always stay above the corresponding at-the-money put option value lines. For the one-shout and two-shout options, $\tau = 12$ is larger than the critical value of the time to expiry beyond which it is never optimal to shout.

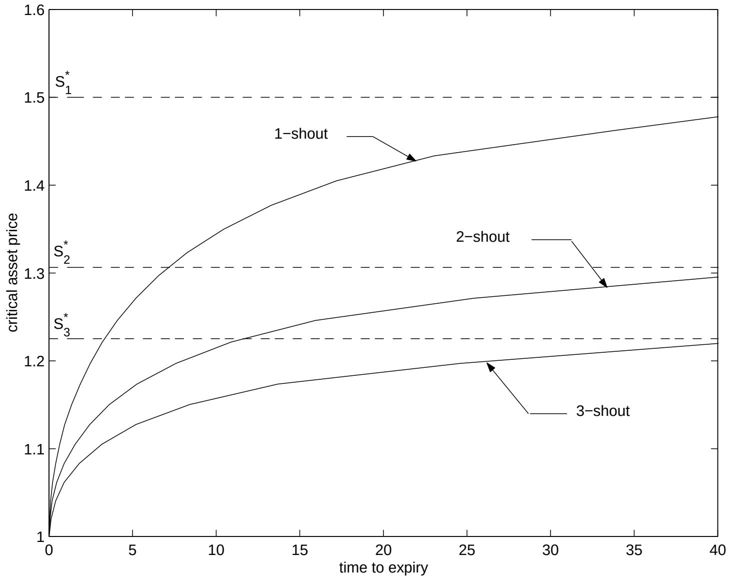

FIGURE 4a. Plot of the shouting boundary as a function of time to expiry for the resettable put options with the right of one, two and three shouts, respectively, given $r < q$. The parameter values used in the calculations are: $r = 0.02, q = 0.06, \sigma = 0.2$ and $X = 1.0$. The asymptotic values of the critical asset price at infinite time to expiry are found to be $1.5, 1.31$ and 1.23, respectively.

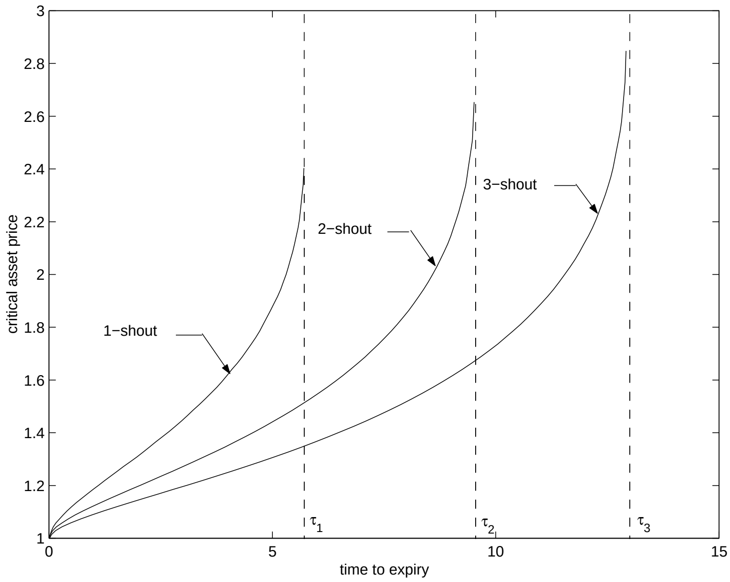

FIGURE 4b. Plot of the shouting boundary as a function of time to expiry for the resettable put options with the right of one, two and three shouts, respectively, given $r > q$. The parameter values used in the calculations are: $r = 0.06, q = 0.02, \sigma = 0.2$ and $X = 1.0$. The critical values of the time to expiry beyond which it is never optimal to shout are estimated to be $5.71, 9.55$ and 13.0, respectively.