Computation of the Shallow Water Equations using the Unified Coordinates.

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Abstract.

Two general coordinate systems have been used extensively in computational fluid dynamics: the Eulerian and the Lagrangian. The Eulerian coordinates cause excessive numerical diffusion across flow discontinuities, slip lines in particular. The Lagrangian coordinates, on the other hand, can resolve slip lines sharply but cause severe grid deformation, resulting in large errors and even breakdown of the computation. Recently, Hui et. al. (W.H. Hui, P.Y. Li and Z.W. Li, *A Unified Coordinate System for Solving the Two-Dimensional Euler Equations*, Journal of Computational Physics, Vol. 153 (1999), pp. 596-637) have introduced a unified coordinate system which moves with velocity $h\mathbf{q}$, $\mathbf{q}$ being velocity of the fluid particle. It includes the Eulerian system as a special case when $h = 0$ and the Lagrangian when $h = 1$, and was shown for the two-dimensional Euler equations of gas dynamics to be superior to both Eulerian and Lagrangian systems. The main purpose of this paper is to adopt this unified coordinate system to solve the shallow water equations. It will be shown that computational results using the unified system are superior to existing results based on either Eulerian system or Lagrangian system in that it **(a)** resolves slip lines sharply, especially for steady flow, **(b)** avoids grid deformation and computation breakdown in Lagrangian coordinates, and **(c)** avoids spurious flow produced by Lagrangian coordinates.

1. Introduction.

Two general coordinate systems have been used extensively for describing fluid motion: the Eulerian and the Lagrangian. Computationally, each system has its advantages as well as disadvantages.

In using the Eulerian coordinates the computational cells are fixed in space, while fluid particles move across cell interfaces in any direction. It is this convective flux that causes excessive numerical diffusion in the numerical solution. Indeed, slip lines are smeared badly and shocks are also smeared, albeit somewhat better than slip lines. Moreover, the smearing of slip lines ever increases with time and distance unless special treatments, such as artificial compression or sub-cell resolution, are employed [2-4] which are, however, not always reliable. Another disadvantage of the Eulerian coordinates is that a grid generation, which can be time-consuming, is needed prior to flow computation in order to satisfy boundary conditions on solid boundaries.

Computational cells in the Lagrangian coordinates, on the other hand, are literally fluid particles. Consequently, there is no convective flux across cell interfaces and numerical diffusion is thus minimized. However, the very fact that the computational cells exactly follow fluid particles can result in severe grid deformation, causing inaccuracy and even breakdown of the computation. To prevent this from happening, the most famous Lagrangian method in use at the present time - the Arbitrary Lagrangian- Eulerian Technique (ALE) [5-7] - uses continuous re-zoning and re-mapping to the Eulerian grid. Unfortunately, this process requires interpolations of geometry and flow variables which result in loss of accuracy, manifested as numerical diffusion which ALE wants to avoid in the first place. Indeed, it was demonstrated in [8] that re-zoning results in diffusive errors of the type encountered in Eulerian solutions and that continuously re-zoned Lagrangian computation is equivalent to an Eulerian computation. Another disadvantage of the Lagrangian coordinates is that, except in the simple case of one-dimensional unsteady flow, the governing equations for inviscid flow are not easily written in conservation form, making it difficult to capture shocks correctly.

Recently, Hui et. al. [1] have introduced a unified coordinate system which moves with velocity $h\mathbf{q}$, where $\mathbf{q}$ is velocity of the fluid particle. It includes the Eulerian coordinates as a special case when $h = 0$ and the Lagrangian when $h = 1$ and, more importantly, it has a new degree of freedom in choosing the arbitrary function h to improve the quality of computational results. In particular, it was shown in [1] that for the two-dimensional Euler equations of gas dynamics, choosing the function h to preserve grid angles results in a coordinate system which is superior to both Eulerian and Lagrangian systems.

The purpose of this paper is to adopt this unified coordinate system to solve the shallow water equations; it will be shown that computational results using the unified system are superior to existing results based on either Eulerian system or Lagrangian system.

The paper is arranged as follows. In Section 2 the shallow water equations in conservation form are derived using the unified coordinates. Sections 3 and 4 study the cases of one-dimensional and two-dimensional flow, respectively. Section 5 gives results for several test examples computed using the unified coordinates and compares them with Eulerian or Lagrangian computation. Finally, conclusions are given in Section 6.

2. Shallow Water Equations in the Unified Coordinates.

The shallow water equations in conservation form using Cartesian coordinates are

$$\frac{\partial}{\partial t}\begin{pmatrix} \zeta \\ \zeta u \\ \zeta v \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \zeta u \\ \zeta u^2 + \frac{1}{2}g\zeta^2 \\ \zeta uv \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \zeta v \\ \zeta uv \\ \zeta v^2 + \frac{1}{2}g\zeta^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta\left[g\frac{\partial D}{\partial x} - m\right] \\ \zeta\left[g\frac{\partial D}{\partial y} - n\right] \end{pmatrix} \quad (1)$$

where g is the acceleration due to gravity, $\zeta(x, y, t)$ is the total water height measured from the bottom, $u(x, y, t)$ and $v(x, y, t)$ are the components of the fluid velocity in the horizontal x and y direction, respectively, D(x,y) is the water depth from a fixed reference level, and m and n are the components of the bottom friction force due to its roughness.

Introduce a transformation of coordinates from (t, x, y) to (λ, ξ, η)

$$\begin{cases} dt = d\lambda \\ dx = hud\lambda + Ad\xi + Ld\eta \\ dy = hvd\lambda + Bd\xi + Md\eta \end{cases} \tag{2}$$

where h is arbitrary. Let

$$\frac{D_h}{Dt} \equiv \frac{\partial}{\partial t} + hu\frac{\partial}{\partial x} + hv\frac{\partial}{\partial y} \tag{3}$$

denote the time derivative following the **pseudo-particle**, whose velocity is $h\mathbf{q}$, $\mathbf{q} = (u, v)$. Then, it is easy to show

$$\frac{D_h \xi}{Dt} = 0, \quad \frac{D_h \eta}{Dt} = 0 \tag{4}$$

that is to say, the coordinates (ξ, η) are material functions of the pseudo-particles, hence are their permanent identifications. Accordingly, computational cells move and deform with pseudo-particles, rather than with fluid particles as in Lagrangian coordinates. The concept of pseudo-particles was first introduced in [1] and it was very successful in computationally resolving flow discontinuities as well as in understanding and interpreting the computational results.

Remarks.

(a) Unlike most transformations used in grid generation, which are flow-independent, the unique feature of transformation (2) is its dependence on the fluid velocity.

(b) In (2), h is an arbitrary function of coordinates (λ, ξ, η). On the other hand, (A, B, L, M) are determined by the compatibility conditions. For example, for dx to be a total differential,

$$\frac{\partial A}{\partial \lambda} = \frac{\partial (hu)}{\partial \xi} \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = \frac{\partial (hu)}{\partial \eta} \tag{6}$$

When (5-6) are satisfied the other compatibility condition, namely

$$\frac{\partial A}{\partial \eta} = \frac{\partial L}{\partial \xi} \tag{7}$$

is also satisfied, provided it is at $\lambda = 0$ which can always be ensured in numerical computation. Similar compatibility conditions hold for (B, M).

(c) In the special case when $h = 0$, (A, B, L, M) are independent of λ. Then the coordinates (ξ, η) are independent of time λ and are hence fixed in space. This coordinate system is thus Eulerian. Transformation (2) is then flow-independent and is just like any other transformation from Cartesian coordinates (x, y) to curvilinear coordinates (ξ, η) used in grid generation. In particular, if $A = M = 1$ and $L = B = 0$, (ξ, η) are identical with Cartesian coordinates (x, y).

(d) In the special case when $h = 1$, on the other hand, the pseudo-particles coincide with fluid particles and (ξ, η) are the material functions of fluid particles, hence are Lagrangian coordinates. The conventional choice of the Lagrangian coordinates, i.e., $(\xi, \eta) = (x, y)|_{t=0}$, is just a special choice of material functions, corresponding to choosing $A = M = 1$ and $L = B = 0$. It does not offer any particular advantage in numerical computation; rather (ξ, η) should better be left to be suitably chosen to initialize numerical computation. In particular, the computational domain in (ξ, η) space can always be easily made regular, e.g. rectangular, even if it is irregular in the physical space. This cannot be done with the conventional choice of the Lagrangian coordinates.

(e) In the general case, h is arbitrary. It has been shown [1] that the unified coordinates for $h \neq 0$ always yield sharp slip line resolution in steady flow. Furthermore, h may be chosen to advantage: to avoid excessive numerical diffusion in the Eulerian coordinates, and/or to avoid severe grid deformation in the Lagrangian coordinates.

Under the transformation (2) the shallow water equations (1) become

$$\frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} = \mathbf{S}, \tag{8}$$

where

$$\mathbf{E} = \begin{pmatrix} \zeta\triangle \\ \zeta\triangle u \\ \zeta\triangle v \\ A \\ B \\ L \\ M \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \zeta(1-h)I \\ \zeta(1-h)Iu + \frac{1}{2}g\zeta^2 M \\ \zeta(1-h)Iv - \frac{1}{2}g\zeta^2 L \\ -hu \\ -hv \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \zeta(1-h)J \\ \zeta(1-h)Ju - \frac{1}{2}g\zeta^2 B \\ \zeta(1-h)Jv + \frac{1}{2}g\zeta^2 A \\ 0 \\ 0 \\ -hu \\ -hv \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 0 \\ \zeta\left[gMD_\xi - gBD_\eta - m\triangle\right] \\ \zeta\left[-gLD_\xi + gAD_\eta - n\triangle\right] \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

with

$$\triangle = AM - BL, \quad I = uM - vL, \quad J = vA - uB \tag{10}$$

We note that the shallow water equations (8) written in the unified coordinates are in conservation form. We also point out that although system (8) is larger than (1), computationally the extra computing time required for solving the last four equations of (8) is very small, typically $3-5\%$, because the bulk of computing time is spent on solving the Riemann problems for the first three equations of (8), which require same amount of computing time as system (1).

In the remainder of this paper we shall consider only horizontal bottom and neglect the friction term there, hence $\mathbf{S} = 0$.

3. One-dimensional Shallow Water Flow.

For the special case of one-dimensional flow, transformation (2) simplifies to

$$\begin{cases} dt = d\lambda \\ dx = hud\lambda + Ad\xi \end{cases} \tag{11}$$

and the shallow water equations (8) become

$$\frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = \mathbf{0}, \tag{12}$$

where

$$\mathbf{E} = \begin{pmatrix} \zeta A \\ \zeta A u \\ A \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \zeta(1-h)u \\ \zeta(1-h)u^2 + \frac{1}{2}g\zeta^2 \\ -hu \end{pmatrix} \tag{13}$$

3.1. Hyperbolicity.

It is well known that the one-dimensional shallow water equations in Eulerian coordinates are hyperbolic. But since transformation (11) involves the dependent variable u, there is no guarantee that the resulting system (12) will necessarily be hyperbolic also; this we now check.

The eigenvalues of (12) can be found by direct computation, and the results are:

$$\sigma_1 = 0 \tag{14}$$

$$\sigma_{\pm} = \frac{(1-h)u \pm \sqrt{g\zeta}}{A}. \tag{15}$$

The corresponding right eigenvectors, when the primitive variables $\mathbf{U} = (\zeta, u, A)^T$ are used, are

$$\mathbf{r_1} = (0, 0, 1)^T \tag{16}$$

$$\mathbf{r}_{\pm} = (1, \pm\sqrt{g/\zeta}, \mp\frac{h}{\sigma_{\pm}}\sqrt{g/\zeta})^T \tag{17}$$

It can be shown easily that the σ_1-field is linearly degenerate, while the $\sigma_{\pm}$-fileds are genuinely nonlinear. The eigenvectors are linearly independent, forming a complete basis in the state space; system (12) is therefore hyperbolic for all values of h, despite the fact that transformation (11) involves the dependent variable u. This includes the Eulerian coordinates as a special case when $h = 0$ and the Lagrangian one when $h = 1$. This result is the same as one-dimensional Euler equations of gas dynamics [9].

3.2. Riemann problem.

The one-dimensional shallow water equations (12) will be solved using the Godunov method with MUSCL update to high resolution, of which the main ingredient is the solution of the Riemann problem.

The Riemann problem is

$$\begin{cases} \frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = 0 & \lambda > 0 \\ \mathbf{E}(0,\xi) = \begin{cases} \mathbf{E}_l, & \xi < 0 \\ \mathbf{E}_r, & \xi > 0 \end{cases} \end{cases} \tag{18}$$

where $\mathbf{E}_l$ and$\mathbf{E}_r$ are the constant vectors representing the flow states on the left and the right side, respectively. Here we shall consider the case when h is a constant in the range $0 \leq h \leq 1$. With $h = const.$, eq.(18) is a system of conservation law equations with constant coefficients and a solution to the Riemann problem depends on $\mu = \frac{\xi}{\lambda}$ alone, i.e. it is a self-similar solution of the form $\mathbf{E} = \mathbf{E}(\mu)$. It consists of at most four uniform flow regions, including $\mathbf{E}_l$ and$\mathbf{E}_r$, separated by three elementary waves: a shock (or expansion), a contact line, and an expansion (or shock). These elementary wave solutions are now given.

3.2.1. Expansion wave.

The centered expansion wave solution from the $\sigma_\pm$ characteristic fields can be derived from the following system of ODEs

$$\frac{du}{d\zeta} = \pm\sqrt{g/\zeta} \tag{19}$$

$$\frac{dA}{d\zeta} = \mp \frac{h\sqrt{g/\zeta}}{\sigma_\pm} \tag{20}$$

The solution for (u, A) relates the flow state $\mathbf{U} = (\zeta, u, A)^T$ in the expansion wave to the initial state $\mathbf{U_0} = (\zeta_0, u_0, A_0)^T$ upstream of the wave through the following expressions

$$u \mp 2\sqrt{g\zeta} = u_0 \mp 2\sqrt{g\zeta_0} \tag{21}$$

$$A = A_0 \left(\frac{\zeta_0 - C_\pm}{\zeta - C_\pm} \right)^{\frac{2h}{3-2h}}, \quad C_\pm = \frac{1-h}{3-2h}(2\sqrt{\zeta_0} \mp \frac{u_0}{\sqrt{g}}). \tag{22}$$

To find the solution inside the expansion wave, we consider the characteristic ray through the origin $(0, 0)$ and a general point (λ, ξ) inside the wave. The slope of the characteristic is

$$\frac{d\xi}{d\lambda} = \frac{\xi}{\lambda} = \mu = \sigma_\pm = \frac{(1-h)u \pm \sqrt{g\zeta}}{A} \tag{23}$$

This, together with (21) and (22), gives $\zeta(\mu)$, $u(\mu)$ and $A(\mu)$ implicitly; in the special case of $h = 0$ or $h = 1$, these functions can be written explicitly.

3.2.2. Shocks.

We start from the Rankine-Hugoniot jump conditions of system (18):

$$s[\zeta A] = [(1-h)\zeta u] \tag{24}$$

$$s[\zeta u A] = [(1-h)\zeta u^2 + \frac{g}{2}\zeta^2] \tag{25}$$

$$s[A] = -[hu] \tag{26}$$

where [.] denotes the jump across the discontinuity whose speed is denoted by $s = \frac{d\xi}{d\lambda}$

We denote the pre-shock flow state by $\mathbf{U_0} = (\zeta_0, u_0, A_0)^T$ and the post shock flow state by $\mathbf{U} = (\zeta, u, A)^T$, respectively. Then the shock jump relations after some algebraic manipulations can be expressed as follows:

$$u = u_0 \pm \sqrt{g(\zeta + \zeta_0)(\zeta - \zeta_0)^2/(2\zeta\zeta_0)} \tag{27}$$

$$A = A_0 - \frac{h(u - u_0)}{s_{sh\pm}} \tag{28}$$

where

$$s_{sh\pm} = \frac{(1-h)u_0}{A_0} \pm \frac{\sqrt{g\zeta(\zeta+\zeta_0)/(2\zeta_0)}}{A_0} \tag{29}$$

Formulas (27)-(28) hold for $h : 0 \leq h \leq 1$.

3.2.3. Contact (or slip) lines.

The degenerate wave corresponds to speed $s = \sigma_1 = 0$. From the Rankine-Hugoniot jump conditions (24) to (26) we have

$$\zeta = \zeta_0 \tag{30}$$

$$u = u_0 \tag{31}$$

The only variable which can change its value across this wave is A. Since the flow variables ζ and u are continuous, **there is no flow discontinuity in the form of a contact (slip) line**.

We compare here the computation of the one-dimensional shallow water flow with the one-dimensional flow of gas dynamics. As seen from the analysis, the one-dimensional shallow water equations have just one type of flow discontinuity - shocks, but there is no flow contact (slip) lines which exist in the one-dimensional Euler equations of gas dynamics. In the latter case it was shown ([9]) that Lagrangian system of coordinates is the best for resolution of the contact lines. But with no contact line to resolve, Eulerian and Lagrangian coordinates are on the same footing for accuracy, as is verified in our computation. The adaptive Godunov scheme ([10,11]), which resolves shocks crisply, can now be applied to either the Lagrangian coordinates or the Eulerian ones, or indeed for any h.

4. Two-dimensional Shallow Water Flow.

4.1. Hyperbolicity.

It is well known that the system of unsteady shallow water equations (1) in Cartesian coordinates is hyperbolic, meaning that all its eigenvalues are real and there exist a complete set of linearly independent eigenvectors. Because the transformation from (t, x, y) to the unified coordinates (λ, ξ, η) involves the dependent variables (u, v), there is no guarantee that the resulting system (8) will necessarily be hyperbolic. We now study the hyperbolicity of the system (8). To do that we re-write the system (8) as

$$\mathbf{A}\frac{\partial \mathbf{U}}{\partial \lambda} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial \xi} + \mathbf{C}\frac{\partial \mathbf{U}}{\partial \eta} = \mathbf{S}_1 \tag{32}$$

where

$$\mathbf{U} = \begin{pmatrix} \zeta \\ u \\ v \\ A \\ B \\ L \\ M \end{pmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ uh_\xi \\ vh_\xi \\ uh_\eta \\ vh_\eta \end{pmatrix} \tag{33}$$

and

$$\mathbf{A} = \frac{\partial \mathbf{E}}{\partial \mathbf{U}}, \quad \mathbf{B} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}, \quad \mathbf{C} = \frac{\partial \mathbf{G}}{\partial \mathbf{U}} \tag{34}$$

System (32) is said to be hyperbolic (also called strongly hyperbolic, or fully hyperbolic) in λ if [12]

(i) all the eigenvalues σ of

$$det(\sigma \mathbf{A} - \alpha \mathbf{B} - \beta \mathbf{C}) = 0$$

are real for every pair $(\alpha, \beta) \in R^2 : \alpha^2 + \beta^2 = 1$; and

(ii) associated with the eigenvalues there exist a complete set of seven linearly independent right eigenvectors in the state space.

System (32) is said to be weakly hyperbolic in λ if (i) is satisfied and (ii) is not.

The eigenvalues of (32) can be found by direct computation, and the results are as follows:

Case (a): $h \neq 1$. In this case we get

$$\sigma_1 = 0 \qquad (multiplicity\ 4) \tag{35}$$

$$\sigma_2 = (1-h)(\alpha' u + \beta' v) \tag{36}$$

$$\sigma_\pm = \sigma_2 \pm \sqrt{g\zeta(\alpha'^2 + \beta'^2)}. \tag{37}$$

where

$$\alpha' = (\alpha M - \beta B)/\triangle, \qquad \beta' = -(\alpha L - \beta A)/\triangle.$$

The corresponding right eigenvectors are

$$\mathbf{r}_1 = (0,0,0,1,0,0,0)^T \tag{38}$$

$$\mathbf{r}_2 = (0,0,0,0,1,0,0)^T \tag{39}$$

$$\mathbf{r}_3 = (0,0,0,0,0,1,0)^T \tag{40}$$

$$\mathbf{r}_4 = (0,0,0,0,0,0,1)^T \tag{41}$$

for σ_1,

$$\mathbf{r}_5 = (0, \beta', -\alpha', -\frac{h}{\mu}\alpha\beta', \frac{h}{\mu}\alpha\alpha', -\frac{h}{\mu}\beta\beta', \frac{h}{\mu}\beta\alpha')^T \tag{42}$$

for σ_2, and

$$\mathbf{r}_{6,7} = (1, \frac{\alpha' g}{m_\pm}, \frac{\beta' g}{m_\pm}, \frac{-\alpha\alpha' gh}{\sigma_\pm m_\pm}, \frac{-\alpha\beta' gh}{\sigma_\pm m_\pm}, \frac{-\beta\alpha' gh}{\sigma_\pm m_\pm}, \frac{-\beta\beta' gh}{\sigma_\pm m_\pm})^T \tag{43}$$

for $\sigma_\pm$, where

$$m_\pm = \pm\sqrt{g\zeta(\alpha'^2 + \beta'^2)}$$

The eigenvectors $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_7$ are linearly independent, forming a complete basis in the state space; system (32) is therefore hyperbolic for $h \neq 1$. This includes the Eulerian coordinates as a special case when $h = 0$.

Case (b): $h = 1$ **(Lagrangian case).** In this case the eigenvalues are

$$\sigma_1 = 0 \qquad (multiplicity\ 5) \tag{44}$$

$$\sigma_\pm = \pm\sqrt{g\zeta(\alpha'^2 + \beta'^2)}. \tag{45}$$

The eigenvectors associated with $\sigma_\pm$ are

$$\mathbf{r}_\pm = (1, \frac{\alpha' g}{\sigma_\pm}, \frac{\beta' g}{\sigma_\pm}, \frac{-\alpha\alpha' g}{\sigma_\pm^2}, \frac{-\alpha\beta' g}{\sigma_\pm^2}, \frac{-\beta\alpha' g}{\sigma_\pm^2}, \frac{-\beta\beta' g}{\sigma_\pm^2})^T \tag{46}$$

Associated with $\sigma_1 = 0$ (multiplicity 5),

$$rank(\sigma \mathbf{A} - \alpha \mathbf{B} - \beta \mathbf{C})\Big|_{\sigma=\sigma_1} = 3;$$

hence there exist four, and only four, linearly independent right eigenvectors:

$$\mathbf{r}_1 = (0,0,0,1,0,0,0)^T \tag{47}$$

$$\mathbf{r}_2 = (0,0,0,0,1,0,0)^T \tag{48}$$

$$\mathbf{r}_3 = (0,0,0,0,0,1,0)^T \tag{49}$$

$$\mathbf{r}_4 = (0,0,0,0,0,0,1)^T. \tag{50}$$

We therefore arrive at the conclusion that **the system of unsteady 2-D shallow water equations in Lagrangian coordinates is weakly hyperbolic**, lacking one eigenvector although all eigenvalues are real. It is interesting to note that the similar results were obtained in the gas dynamics case, namely, the two-dimensional and three-dimensional Euler equations written in the unified coordinates are hyperbolic for any h, $0 \leq h < 1$, but they are only weakly hyperbolic for $h = 1$ (Lagrangian case).

In summary, use of Lagrangian coordinates in CFD for two-dimensional shallow water equations not only can cause severe grid deformation, but also renders the two-dimensional shallow water equations weakly hyperbolic, with all its possible consequences on numerical computation. More will be said about the Lagrangian case in section 4.4 below.

4.2. Determination of *h*.

As mentioned earlier, the chief advantage of the unified coordinates is the new degree of freedom in choosing h. Many choices are possible and the simplest one would be to choose a constant value for it, as was done in section 3. Numerical experiments for constant h will be presented in section 6 to show its effects on grid deformation and on resolution of flow discontinuities. In general, it is necessary to restrict h to within the range $0 \le h \le 1$. For $h > 1$, the eigenvalue σ_2 in (36) has an opposite sign to that for $h < 1$, indicating signals propagate in the wrong direction. Our computations for $h > 1$ break down immediately. On the other hand, for $h < 0$, which means the pseudo-particles are moving in the opposite direction to the fluid particles, computation can be carried out initially but after some finite time it breaks down also. No difficulty has been encountered in all our computations if h is restricted to $0 \le h < 1$.

As shown in [1], a good choice for h is to preserve the grid angles in the solution process which marches in λ, i.e.

$$\frac{\partial}{\partial \lambda}\left[\frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{\nabla \eta}{|\nabla \eta|}\right] = 0 \tag{51}$$

Since

$$\begin{aligned} \nabla \xi &= (M, -L)/\triangle \\ \nabla \eta &= (-B, A)/\triangle \end{aligned} \tag{52}$$

condition (51) becomes

$$\frac{\partial}{\partial \lambda}\left[\frac{AL + BM}{\sqrt{A^2 + B^2}\sqrt{L^2 + M^2}}\right] = 0 \tag{53}$$

By making use of the last four equations of (8), it is easy to show that (53) is equivalent to

$$S^2 J \frac{\partial h}{\partial \xi} + T^2 I \frac{\partial h}{\partial \eta} = \left[S^2 \left(B\frac{\partial u}{\partial \xi} - A\frac{\partial v}{\partial \xi}\right) - T^2 \left(M\frac{\partial u}{\partial \eta} - L\frac{\partial v}{\partial \eta}\right)\right] h \tag{54}$$

where

$$S^2 = L^2 + M^2, \quad T^2 = A^2 + B^2 \tag{55}$$

A consequence of determining h from (54) is that if the grid is orthogonal at $\lambda = 0$ it will remain so for subsequent λ. Orthogonal grid is known to possess many desirable properties over non-orthogonal grids, e.g. attaining higher accuracy than non-orthogonal grids. Computationally, Eq. (54) is to be solved at every time step after the flow variables $\mathbf{Q} = (\zeta, u, v)^T$ and the geometric variables $\mathbf{K} = (A, B, L, M)^T$ are found. It is thus a first order linear partial differential equations for $h(\xi, \eta; \lambda)$, with λ appearing as a parameter. To find solution h in the range

$$0 \le h \le 1 \tag{56}$$

we note that (54) is linear and homogeneous, therefore it possesses two properties: (a) positive solution $h > 0$ always exists, and (b) if h is a solution to (54) so is h/C, C being any constant. Making use of property (a), we let $g = \ln(hq)$ to get

$$\begin{aligned} & S^2(A\sin\theta - B\cos\theta)\frac{\partial g}{\partial \xi} + T^2(M\cos\theta - L\sin\theta)\frac{\partial g}{\partial \eta} \\ = \; & S^2\left(B\frac{\partial \cos\theta}{\partial \xi} - A\frac{\partial \sin\theta}{\partial \xi}\right) - T^2\left(M\frac{\partial \cos\theta}{\partial \eta} - L\frac{\partial \sin\theta}{\partial \eta}\right) \end{aligned} \tag{57}$$

where $q = \sqrt{u^2 + v^2}$ and θ is the flow angle: $u = q\cos\theta, v = q\sin\theta$. Now, if g_1 is any solution to (57) then $h = e^{g_1}/qC$ is a solution to (54) satisfying condition (56), provided we choose C equal to the maximum of e^{g_1}/q over the whole flow field being computed. The reason to work with $\ln(hq)$ instead of $\ln h$ is that from our experience with steady flow [13], hq is continuous across slip lines, hence working with hq can minimize the numerical errors.

Numerically, Eq. (57) is solved easily by the method of characteristics if their slopes do not change sign; otherwise it is solved by iteration.

4.3. Solution trategies.

As the system of shallow water equations (8) written in unified coordinates is in conservation form, any well-established shock-capturing method can be used to solve it. We shall use the Godunov method with the MUSCL update to higher resolution to solve system (8). The computation will be done entirely in the $\lambda - \xi - \eta$ space. A physical cell in the $x - y$ plane marching along the pseudo-particle's pathline corresponds to a rectangular cell in the $\xi - \eta$ plane marching in the λ direction in the computational space $\lambda - \xi - \eta$.

The superscript k refers to the marching time step number and the subscripts i and j refer to the cell index number on a time plane $\lambda = const$. The time step $\Delta\lambda^k = \lambda^{k+1} - \lambda^k$ is uniform for all i and j, but is always chosen to satisfy the CFL stability condition. The grid divides the computational domain into cubic control volumes, or cells, which in ξ and η direction are centered at $(\lambda^k, \xi_i, \eta_j)$ and have widths $\Delta\xi_i = \xi_{i+1/2} - \xi_{i-1/2}$ and $\Delta\eta_j = \eta_{j+1/2} - \eta_{j-1/2}$ (for all k). Unless otherwise stated we shall use uniform cell width $\Delta\xi_i$ for all i and $\Delta\eta_j$ for all j.

In the physical space (t, x, y) a cuboid cell marching in (λ, ξ, η) space corresponds to a pseudo-particle marching along its path tube with step Δt ($\Delta t = \Delta\lambda$). The pseudo-particle is bounded by four path surfaces $\xi = \xi_{i\pm1/2}$ and $\eta = \eta_{j\pm1/2}$ around it. Initially, any curvilinear coordinate grid on the $x - y$ plane may be used as the $\xi - \eta$ coordinate grid and the initial geometric variables $\mathbf{K} = (A, B, L, M)^T$ can be determined from (2) as part of the initial conditions. A stationary solid wall is always a path surface of the fluids and hence also of the pseudo-fluids; it is therefore a coordinate surface of the unified coordinates.

Applying the divergence theorem to (8) over the cuboid cell (i, j, k) results in

$$\mathbf{E}_{i,j}^{k+1} = \mathbf{E}_{i,j}^{k} - \frac{\Delta\lambda^k}{\Delta\xi_i}\left(\mathbf{F}_{i+1/2,j}^{k+1/2} - \mathbf{F}_{i-1/2,j}^{k+1/2}\right) - \frac{\Delta\lambda^k}{\Delta\eta_j}\left(\mathbf{G}_{i,j+1/2}^{k+1/2} - \mathbf{G}_{i,j-1/2}^{k+1/2}\right), \tag{58}$$

$$i = 1, 2, ..., m; \quad j = 1, 2, ..., n, \tag{59}$$

where the notation for the cell-average of any quantity f is

$$f_{i,j}^k = \frac{1}{\Delta\xi_i\Delta\eta_j}\int_{\xi_{i-1/2}}^{\xi_{i+1/2}}\int_{\eta_{j-1/2}}^{\eta_{j+1/2}} f(\lambda^k, \xi, \eta)d\xi\, d\eta, \tag{60}$$

and the notation for λ-average of f is:

$$f_{i+1/2,j}^{k+1/2} = \frac{1}{\Delta\lambda^k}\int_{\lambda^k}^{\lambda^{k+1}} f(\lambda, \xi_{i+1/2}, \eta_j)d\lambda, \tag{61}$$

$$f_{i,j+1/2}^{k+1/2} = \frac{1}{\Delta\lambda^k}\int_{\lambda^k}^{\lambda^{k+1}} f(\lambda, \xi_i, \eta_{j+1/2})d\lambda. \tag{62}$$

According to Godunov's idea, the cell-interface fluxes $\mathbf{F}_{i+1/2,j}^{k+1/2}$ and $\mathbf{G}_{i,j+1/2}^{k+1/2}$ for the cell (i, j) are to be obtained from the self-similar solution of a local two-dimensional

Riemann problem formed by the averaged constant state $\mathbf{E}_{i,j}$ of the cell (i,j) and those of its adjacent cells. Unfortunately, such a solution to (8) is unavailable at present time. Indeed, even a 2-D Riemann solution to the simpler system (1), which is a special case of (8) when $h=0$, is not yet available. On the other hand, it is known that a monotone difference scheme to a general conservation form converges to the physically relevant entropy-satisfying solution. In particular, Crandall and Majda [14] proved convergence for dimensional splitting algorithms when each step is approximated by a monotone difference scheme (such as the Godunov scheme) for a scalar conservation law in multi-dimension.

In view of the above, we shall numerically solve (8) using a Godunov-type scheme based on the dimensional splitting approximation to reduce the two-dimensional flow problem to two one-dimensional flow problems.

The dimensional splitting technique for finding an approximate solution to the Riemann problem in multi-dimensional flow is now well established and used widely. This technique renders the solution of a multi-dimensional problem to a sequential solution of several one-dimensional problems. The Godunov splitting and the Strang splitting [15] are frequently used in practical applications. We shall use the Strang splitting in this paper. Let $\mathcal{L}^{\xi}_{\Delta\lambda}$ represent the exact solution operator for the 1-D equation in $\lambda-\xi$ plane and $\mathcal{L}^{\eta}_{\Delta\lambda}$ similarly defined, then according to Strang splitting

$$\mathbf{E}^{k+1}=\mathcal{L}^{\xi}_{\frac{\Delta\lambda}{2}}\mathcal{L}^{\eta}_{\Delta\lambda}\mathcal{L}^{\xi}_{\frac{\Delta\lambda}{2}}\mathbf{E}^{k} \tag{63}$$

where $\Delta\lambda=\lambda^{k+1}-\lambda^{k}$.

The solution operator $\mathcal{L}^{\xi}_{\Delta\lambda}$ for the Riemann problem with variable coefficients in the governing equations in $\lambda-\xi$ plane will now be given in details.

4.4. The Riemann problem resulting from dimensional splitting of the two-dimensional shallow water equations.

With the use of dimensional splitting, the solution of the original two-dimensional equation system is replaced by the sequential computation involving two one-dimensional equation systems. We consider in details the resulting Riemann problem in the λ - ξ plane; that in the λ - η plane can be discussed similarly.

In the $\mathcal{L}^\xi$ operator, it is assumed that $\frac{\partial}{\partial \eta} = 0$. Hence (8) becomes

$$\frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = 0 \tag{64}$$

where

$$\mathbf{E} = \begin{bmatrix} \zeta\triangle \\ \zeta\triangle u \\ \zeta\triangle v \\ A \\ B \\ L \\ M \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \zeta(1-h)I \\ \zeta(1-h)Iu + \frac{1}{2}g\zeta^2 M \\ \zeta(1-h)Iv - \frac{1}{2}g\zeta^2 L \\ -hu \\ -hv \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

with

$$\triangle = AM - BL, \qquad I = uM - vL \tag{66}$$

We rewrite (64) by using the component of velocity $\mathbf{q}$ in the direction $\mathbf{n}$ normal to, and $\mathbf{t}$ tangential to the plane $\xi = const.$, i.e.

$$\mathbf{n} = \frac{\nabla\xi}{|\nabla\xi|} = (M, -L)/S, \quad \mathbf{t} = (L, M)/S \tag{67}$$

$$\omega = \mathbf{q}\cdot\mathbf{n} = (uM - vL)/S \tag{68}$$

$$\tau = \mathbf{q}\cdot\mathbf{t} = (uL + vM)/S \tag{69}$$

where

$$S = (L^2 + M^2)^{1/2} \tag{70}$$

Hereafter we shall abandon the last two equations of system (64), keeping in mind that $L = L(\xi)$ and $M = M(\xi)$ are given.

Equations (64) now become

$$\frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = \mathbf{S}_2 \tag{71}$$

where

$$\mathbf{E} = \begin{bmatrix} \zeta\triangle \\ \zeta\triangle\omega \\ \zeta\triangle\tau \\ A \\ B \end{bmatrix}, \quad \mathbf{F} = S \begin{bmatrix} \zeta(1-h)\omega \\ \zeta(1-h)\omega^2 + \frac{1}{2}g\zeta^2 \\ \zeta(1-h)\omega\tau \\ -h(M\omega + L\tau)/S^2 \\ h(L\omega - M\tau)/S^2 \end{bmatrix}, \quad \mathbf{S}_2 = \frac{\partial\psi}{\partial\xi} S \begin{bmatrix} 0 \\ \zeta(1-h)\omega\tau \\ \zeta(1-h)\omega^2 + \frac{1}{2}g\zeta^2 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

where

$$tan\ \psi = M/L \tag{73}$$

Equations (71) in the $\lambda-\xi$ plane resulting from dimensional splitting of the two-dimensional equations (8) are more complicated than the genuinely one-dimensional equations (12), and the solution to their Riemann problem is explained below.

The Riemann problem resulting from dimensional splitting in the unified coordinates in λ - ξ plane is now

$$\begin{cases} \frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = \mathbf{S}_2 & \lambda > 0 \\ \mathbf{E}(0,\xi) = \begin{cases} \mathbf{E}_l, & \xi < 0 \\ \mathbf{E}_r, & \xi > 0 \end{cases} \end{cases} \tag{74}$$

where $\mathbf{E}_l$ and $\mathbf{E}_r$ are constant, and our purpose is to find the flux $\mathbf{F}$ on $\xi = 0$ to be used in the Godunov scheme to update the conserved quantities $\mathbf{E}$. At time level λ^k (taken to be equal to 0) h_r and h_l are the values of h at the cell $(i+1, j)$ and (i, j). They are **assumed** constant for $0 \le \lambda < \triangle\lambda$, i.e.

$$\frac{\partial h}{\partial \lambda} = 0 \qquad\qquad (0 \le \lambda < \triangle\lambda) \tag{75}$$

and this is consistent with the h - equation (54). But h changes its value at $\lambda = \triangle\lambda$ as given by (54), whose coefficients are evaluated at $\triangle\lambda$.

Now we first find all possible solutions to (71) for $\xi > 0$ and $\xi < 0$ separately, and then use them to construct solution to the Riemann problem (74).

Case (1): $\xi > 0$

Since $L = L_r = const.$ and $M = M_r = const.$, hence $S = S_r = const.$, $\psi = \psi_r = const.$, therefore

$$\mathbf{S}_2 = 0 \tag{76}$$

and

$$\begin{cases} \frac{\partial \mathbf{E}}{\partial \lambda} + \frac{\partial \mathbf{F}}{\partial \xi} = 0 & \lambda > 0, \quad \xi > 0 \\ \mathbf{E}(0, \xi) = \mathbf{E}_r & \xi > 0 \end{cases} \tag{77}$$

where

$$\mathbf{E} = \begin{bmatrix} \zeta\triangle \\ \zeta\triangle\omega \\ \zeta\triangle\tau \\ A \\ B \end{bmatrix}, \quad \mathbf{F} = S \begin{bmatrix} \zeta(1-h)\omega \\ \zeta(1-h)\omega^2 + \frac{1}{2}g\zeta^2 \\ \zeta(1-h)\omega\tau \\ -h(M\omega + L\tau)/S^2 \\ h(L\omega - M\tau)/S^2 \end{bmatrix} \tag{78}$$

with $h = h_r = const.$. Similarity solution to (77) exists in the form

$$\mathbf{E} = \mathbf{E}(\mu), \qquad \mu = \frac{\xi}{\lambda} \tag{79}$$

Possible solutions to (77) are:

(i) a constant state: $\mathbf{E} = const.$;

(ii) a centered expansion wave;

(iii) a shock;

(iv) a contact (slip) line.

Let us first find the eigenvalues and the corresponding right eigenvectors of the system (77).

For smooth solution system (77) can be written as

$$\mathbf{A}\frac{\partial \mathbf{U}}{\partial \lambda} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial \xi} = \mathbf{0} \tag{80}$$

where $\mathbf{U} = (\zeta, \omega, \tau, A, B)^T$, $\mathbf{A} = \frac{\partial \mathbf{E}}{\partial \mathbf{U}}$ and $\mathbf{B} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$.

In order to obtain the eigenvalues σ we need to find the determinant of the matrix $(\sigma\mathbf{A} - \mathbf{B})$. Direct computation gives

$$det(\sigma\mathbf{A} - \mathbf{B}) = \sigma^2 m\zeta^2(m^2 - S^2\zeta g), \quad m = \sigma\triangle - S(1-h)\omega \tag{81}$$

Case (1a): $h \neq 1$. From the vanishing of this determinant we get our eigenvalues.

$$\begin{aligned} \sigma_1 &= 0 \qquad (\textit{multiplicity } 2) &(82)\\ \sigma_2 &= \frac{S}{\triangle}(1-h)\omega &(83)\\ \sigma_\pm &= \frac{S}{\triangle}\{(1-h)\omega \pm \sqrt{g\zeta}\} &(84)\end{aligned}$$

We emphasize again that $S = S_r = const.$ and $h = h_r = const.$, $h \neq 1$. The corresponding set of right eigenvectors is

$$\begin{aligned} \mathbf{r}_1 &= (0,0,0,0,1)^T &(85)\\ \mathbf{r}_2 &= (0,0,0,1,0)^T &(86)\\ \mathbf{r}_3 &= (0,0,1,-\frac{hL}{S\sigma_2},-\frac{hM}{S\sigma_2})^T &(87)\\ \mathbf{r}_\pm &= (1,\pm\sqrt{\frac{g}{\zeta}},0,\mp\frac{hM}{S\sigma_\pm}\sqrt{\frac{g}{\zeta}},\pm\frac{hL}{S\sigma_\pm}\sqrt{\frac{g}{\zeta}})^T &(88)\end{aligned}$$

Since the eigenvectors (85-88) are linearly independent, system (77) is hyperbolic. To classify the characteristic fields, we see that

$$\nabla\sigma_1 \cdot \mathbf{r}_{1,2} = 0 \tag{89}$$

and

$$\nabla\sigma_2 \cdot \mathbf{r}_3 = 0, \tag{90}$$

implying that characteristic fields corresponding to the $\sigma_{1,2}$ and σ_3 are linearly degenerate. On the other hand,

$$\nabla\sigma_\pm = (\pm\frac{S}{2\triangle}\sqrt{\frac{g}{\zeta}},\frac{S(1-h)}{\triangle},0,-\frac{\sigma_\pm M}{\triangle},\frac{\sigma_\pm L}{\triangle}) \tag{91}$$

and

$$\nabla\sigma_\pm \cdot \mathbf{r}_\pm = \pm\frac{3S}{2\triangle}\sqrt{\frac{g}{\zeta}} \neq 0. \tag{92}$$

Therefore the $\sigma_\pm$ characteristic families are genuinely nonlinear.

Case (1b): $h = 1$ **(Lagrangian case).** In this case the eigenvalues are

$$\sigma_1 = 0 \qquad (multiplicity\ 3) \tag{93}$$

$$\sigma_{\pm} = \pm\frac{S}{\triangle}\sqrt{g\zeta}. \tag{94}$$

The eigenvectors associated with $\sigma_{\pm}$ are

$$\mathbf{r}_{\pm} = (1, \pm\sqrt{\frac{g}{\zeta}}, 0, -\frac{M\triangle}{S^2\zeta}, \frac{L\triangle}{S^2\zeta})^T \tag{95}$$

Associated with $\sigma_1 = 0$ (multiplicity 3),

$$rank(\sigma\mathbf{A} - \mathbf{B})\Big|_{\sigma=\sigma_1} = 3;$$

hence there exist two, and only two, linearly independent right eigenvectors:

$$\mathbf{r}_1 = (0, 0, 0, 1, 0)^T \tag{96}$$

$$\mathbf{r}_2 = (0, 0, 0, 0, 1)^T. \tag{97}$$

We therefore arrive at the conclusion that **system (77) resulting from dimensional splitting of the two-dimensional equations in Lagrangian coordinates is weakly hyperbolic**, lacking one eigenvector although all eigenvalues are real. This is in direct contrast to the genuinely one-dimensional flow case, Eq.(12), which is hyperbolic for all values of h, including the Lagrangian case ($h = 1$). Despite this defect, most of our computations with $h = 1$ encounter no difficulty and produce results almost identical to that for $h = 0.999$, for which (77) is hyperbolic. But this is not guaranteed, and we shall present computational results for the case $h = 0.999$ instead of $h = 1$.

We shall now give solutions to the elementary waves in details: the expansion wave, the shock wave and the slip line. These solutions will be used in constructing the Riemann solution to (74).

4.4.1. Expansion wave.

The expansion wave is a smooth solution from the $\sigma_\pm$ characteristic fields which can be derived from the following system of ODEs.

$$\frac{d\omega}{d\zeta} = \pm\sqrt{\frac{g}{\zeta}} \tag{98}$$

$$\frac{d\tau}{d\zeta} = 0 \tag{99}$$

$$\frac{dA}{d\zeta} = \mp\frac{hM}{S\sigma_\pm}\sqrt{\frac{g}{\zeta}} \tag{100}$$

$$\frac{dB}{d\zeta} = \pm\frac{hL}{S\sigma_\pm}\sqrt{\frac{g}{\zeta}} \tag{101}$$

The solution for (ω, τ) relates the flow state $\mathbf{Q} = (\zeta, \omega, \tau)^T$ in the expansion fan to the initial state $\mathbf{Q_0} = (\zeta_0, \omega_0, \tau_0)^T$ upstream of the fan through the following expressions

$$\omega = \omega_0 \mp 2(\sqrt{g\zeta_0} - \sqrt{g\zeta}) \tag{102}$$

$$\tau = \tau_0 \tag{103}$$

Note that on crossing the expansion fan, these relations are independent of $\mathbf{K}_r$ and h_r.

To find the solution inside the expansion fan, we consider the characteristic ray through the origin $(0,0)$ and a general point (λ, ξ) inside the fan. The slope of the characteristic is

$$\frac{d\xi}{d\lambda} = \frac{\xi}{\lambda} = \sigma_\pm = \frac{S}{\triangle}\{(1-h)\omega \pm \sqrt{g\zeta}\} \tag{104}$$

Using the above expression and the equation for ω in (102), we get

$$\zeta = \frac{1}{g(3-2h)^2}\left[(1-h)(\omega_0 \mp 2\sqrt{g\zeta_0}) - \frac{\triangle}{S}\frac{\xi}{\lambda}\right]^2 \tag{105}$$

where $\triangle = AM - BL$ is found from (100) and (101) to be a function of ζ:

$$\triangle = \triangle_0\left(\frac{\zeta_0 - C_\pm}{\zeta - C_\pm}\right)^{\frac{2h}{3-2h}}, \quad C_\pm = \frac{1-h}{3-2h}(2\sqrt{\zeta_0} \mp \frac{\omega_0}{\sqrt{g}}). \tag{106}$$

Eq.(105) together with (106) thus defines an implicit function $\zeta(\mu)$, $\mu = \frac{\xi}{\lambda}$. Eq.(106) reduces to the 1-D case, Eq.(22), when $L = 0$ and $M = 1$. The expressions for ω and τ in terms of μ can be obtained simply via (102-103). Like ζ, they depend on $(\mathbf{K}_r, h_r)$, but at $\mu = \frac{\xi}{\lambda} = 0$ they depend only on h_r. The variations of A and B across an expansion fan can also be obtained from (100) and (101), but they are not needed in calculating the flux (the flux function $\mathbf{F}$ does not involve A or B; it involves only L and M).

4.4.2. Shock waves.

From the Rankine-Hugoniot jump conditions of the system (77), we get

$$
\begin{aligned}
s[\zeta\triangle] &= [\zeta(1-h)\omega S] && (107)\\
s[\zeta\triangle\omega] &= [(\zeta(1-h)\omega^2 + \frac{g\zeta^2}{2})S] && (108)\\
s[\zeta\triangle\tau] &= [\zeta(1-h)\omega\tau S] && (109)\\
s[A] &= -[\frac{h}{S}(M\omega + L\tau)] && (110)\\
s[B] &= [\frac{h}{S}(L\omega - M\tau)] && (111)
\end{aligned}
$$

where [.] denotes the jump across the discontinuity whose speed is denoted by $s = \frac{d\xi}{d\lambda}$. The shock jump relations after some algebraic manipulations can be expressed as follows:

$$
\begin{aligned}
s &= \frac{S}{\triangle_0}\left[(1-h)\omega_0 \pm \sqrt{\frac{g}{2}\frac{\zeta}{\zeta_0}(\zeta+\zeta_0)}\right] && (112)\\
\omega &= \omega_0 \pm \sqrt{\frac{g(\zeta+\zeta_0)(\zeta-\zeta_0)^2}{2\zeta\zeta_0}} && (113)\\
\tau &= \tau_0. && (114)
\end{aligned}
$$

We note that the relations of the flow variables (ζ, ω, τ) across the shock are independent of $\mathbf{K}_r$ and h_r, but the slope of the shock wave s is dependent on $\mathbf{K}_r$ and h_r. But this dependence is not needed in finding the water height ζ, and hence (ω, τ) also, at $\mu = 0$ provided only that $s > 0$. (If $s < 0$, the shock wave will appear in the quadrant ($\xi < 0$, $\lambda > 0$)).

Note that the jumps of A and B across a shock may also be obtained from (110) and (111), but they are not needed in calculating the flux $\mathbf{F}$ at $\mu = 0$.

4.4.3. Slip lines.

For the slip line corresponding to the speed $s = \sigma_3 = \frac{S}{\triangle_0}(1-h)\omega_0 > 0$ we find, from

Rankine-Hugoniot jump conditions (107-111),

$$\zeta = \zeta_0 \tag{115}$$

$$\omega = \omega_0 \tag{116}$$

but τ and A, B may jump arbitrarily.

We note again that, except the speed s, the relations (115-116) across a slip line are independent of $(\mathbf{K}_r, h_r)$. Although s depends on $(\mathbf{K}_r, h_r)$, the dependence is not needed in calculating (ζ, ω, τ) and the flux $\mathbf{F}$ at $\mu = 0$, provided $s > 0$. (If $s < 0$, the slip line appears in the quadrant ($\xi < 0$, $\lambda > 0$)).

The flow across the slip line corresponding to $s = \sigma_{1,2} = 0$ cannot be discussed within the quadrant ($\xi > 0$, $\lambda > 0$) alone.

Case (2): $\xi < 0$.

The solution in the quadrant ($\xi < 0$, $\lambda > 0$) can be obtained similarly.

Case (3): Riemann solution in $-\infty < \xi < +\infty$.

Now, after obtaining all possible solutions for $\xi > 0$ and $\xi < 0$ separately the question is how to construct the solution to the Riemann problem for $\lambda > 0$, $-\infty < \xi < +\infty$. We note that at $\xi = 0$: **a)** $\mathbf{S}_2$ in (71) is a delta function; **b)** the coefficients in $\mathbf{E}$ and $\mathbf{F}$ jump discontinuously. These are the difficulties one would face with within the Eulerian system using curvilinear coordinates rather than cartesian coordinates.

The σ_1-field is linearly degenerate and it corresponds to $s = 0$ in (107-111). These are five equations relating three jumps of ζ, ω and τ and, therefore, in general have no solution, except when $h_r = h_l$, $L_r = L_l$ and $M_r = M_l$. In the latter case, there is a unique solution: the trivial solution $[\zeta] = [\omega] = [\tau] = 0$, i.e. a continuous solution.

To avoid the difficulty of non-existence of solution to the Rankine-Hugoniot relations (107-111), we replace both h_l and h_r by their average, i.e. $h_l = h_r = 0.5(h_l + h_r) = \overline{h}$, and similarly replace L_l and L_r, M_l and M_r by their averages $\overline{M}$ and $\overline{L}$, respectively. Consequently, the Rankine-Hugoniot relations are satisfied and the flow is continuous across $\mu = \frac{\xi}{\lambda} = 0$. We note from previous discussions that these replacements do not alter the relations of the flow variables (ζ, ω, τ) across the elementary waves: (the expansion fan, the shock, and the slip line) as they do not depend on $(\mathbf{K}, h)$. It should be pointed out that the replacements of L_l and L_r by $\overline{L}$ and M_l and M_r by $\overline{M}$ are a fictitious one - they are used only to ensure the existence of solution to (107-111) - but these average values are never used in the computation. On the other hand, the replacement of h_l and h_r by $\overline{h}$ is a real one: it is used in equation (105) when the line $\mu = \frac{\xi}{\lambda} = 0$ is inside the expansion fan which is, however, a rare case.

The Riemann solution for $-\infty < \xi < +\infty$ can now be constructed in the usual way as if the slip line corresponding to $s = \sigma_{1,2} = 0$ did not exist: shock (or expansion fan), slip line, expansion fan (or shock), separated by uniform flow regions.

5. Test Examples

In this section the unified coordinates approach is tested numerically on several examples. The flat bottom case is considered with zero roughness coefficients m and n. In all cases the effects of h on the computational robustness and accuracy are discussed

Example 1. The first problem is purely one dimensional two-dam break problem. In a long channel three different heights of still water are separated by two dams, of which one is located at $x = 0.8$ and the other at $x = 1.2$ (Fig.1a). At $t = 0$ the dam located at $x = 0.8$ is broken instantly and completely, resulting in an expansion wave moving upstream and a bore (shock) rushing downstream (Fig.1b,c,d). The bore (shock) then reaches the second dam ($x = 1.2$) at some time later, triggering it to break completely and resulting in a stronger bore (shock) moving downstream and another expansion wave moving upstream (Fig.1e,f,g). It is difficult for ordinary shock-capturing schemes to accurately compute this problem due to the interaction of the first bore (resulting from the rapture of the first dam) with the second dam. All conventional numerical schemes([16,17,18]) smear the first bore, thus giving only its approximate location (Figs. 4,5). Consequently, it is impossible to determine the exact time of breaking of the second dam. In our computation the shock-adaptive Godunov method ([10,11]), which gives infinite bore resolution, is applied and this difficulty is avoided. The time when the second dam breaks is found to be 0.2375

Figures 1 and 2 show the evolution of water height and velocity with time. We note that after the breaking of the second dam, the water velocity increases significantly (Fig. 2d,e,f). This combination of the strong bore moving with high velocity can be potentially destructive. Comparisons between exact solutions and computed ones at the time $t = 0.1$ (after the first dam broke) and the time $t = 0.3375$ (0.1 time units after the second dam broke) are presented on the Figure 3. Figure 4 presents computed results for water height and velocity at $t = 0.22$ using ordinary Godunov scheme with MUSCL update. We note that smeared shock reaches the second dam prematurely. It is seen in the Figure 5 that with ordinary shock-capturing method the shock is smeared, though its average location is correct, as expected. It is also seen that the shock-adaptive Godunov scheme perfectly resolves the bore, giving its exact location. In all computations we use $h = 0$ (Eulerian).

Example 2. This example, known as Salzman problem in gas dynamics, is presented to show the role an irregular grid plays in producing spurious flow. The problem consists of a rectangular channel (Fig.6) whose walls form reflective boundaries. The initial data are

$$\begin{cases} \zeta = 4.0, \qquad u = 7.4246 \qquad v = 0.0 \qquad\qquad x < 0 \\ \zeta = 1.0, \qquad u = 0.0 \qquad v = 0.0 \qquad\qquad 0 \le x \le 1 \end{cases}$$

They are chosen in such way that the discontinuity, initially located at $x = 0.0$, will result in a plane shock propagating at the speed equal to 9.899 in the x direction. The initial grid has 10 uniform cells in the y-direction ($0.0 \le y \le 0.1$) and 100 non-uniform cells in the x-direction ($0 \le x \le 1$). The coordinates of the cell centers are (Fig.6)

$$\begin{aligned} x_{i,j} &= (i-1) * \triangle + (11-j) * \triangle * sin\frac{\pi(i-1)}{100} \\ y_{i,j} &= (j-1) * \triangle + 0.5 * \triangle \\ i &= 1, ..., 101 \quad j = 1, ..., 10 \end{aligned}$$

where $\triangle = 1/100$. We shall perform computation for $h = 0.999$ and grid-angle preserving h. Figure 7 presents resulting grid (7a) and water height (7b) for the case $h = 0.999$ at the time $t = 0.06$, which is several time steps before the computation breaks down. As we can see this grid is highly deformed near the position $x = 0.5$ behind the shock. Looking at the velocity vectors (Fig.7c) we see that there is a presence of spurious flow: the y-component of velocity (especially near $x = 0.5$) and hence vorticity which, in turn, affects the grid and the water height (Fig.7a,b).

Similar results of spurious flow production were reported by Dukowicz and Meltz in [19] where it was found that Lagrangian coordinates do not preserve the one-dimensionality of a plane shock but produce spurious vorticity. They successfully introduced a technique to filter out the spurious vorticity, while retaining the real one, if present. We have found that the spurious flow can be avoided automatically by using unified coordinates with grid-angle preserving h. The results are presented in Fig.8. Here we note that although the vertical component of velocity is still present (Fig.8c), especially near the shock, it is very small and is not magnified during the computation. These results (Fig.8) are almost identical to those of gas dynamics case in [19], which requires a special technique to filter out the spurious vorticity.

Example 3. The next example is a two-dimensional steady Riemann problem generated by two uniform parallel flows as

$$(\zeta, F, \theta) = \begin{cases} (1, 4, 0) & y > 0.5 \\ (2, 2.4, 0) & y < 0.5 \end{cases}$$

where F is the Froude number and θ the flow angle, $\theta = \tan^{-1}(v/u)$. The flow contains a shock wave, a slip line and an expansion wave (Fig. 9). The slip line is sensitive to the dissipative property of the numerical methods. In [20,21] the problem was solved numerically using a generalized Lagrangian method which perfectly resolves slip lines. However, that method is valid only for steady flow, whereas the method in this paper is valid for unsteady flow as well. Since the analytical solution for the problem is available, it is an excellent benchmark problem for the verification of numerical methods. In the computation, the steady flow is achieved by time marching until the flow structure and the variables do not change with time. A grid of 60×100 with $\Delta\xi = \Delta\eta = 0.01$ is employed in the computation. Initially, a grid with $\Delta x = \Delta y = 0.01$ in the physical plane is laid over a domain of $\{0 \leq x \leq 0.6, 0 \leq y \leq 1\}$. The initial data are given at each cell according to its position in $y > 0.5$ or $y < 0.5$, representing cell-averaged values. The physical domain will change with time according to the pseudo-particle's velocity $h\mathbf{q}$ if h is not zero. If we follow the computational cells (pseudo-particles), they will move out the initial physical domain, and it would be difficult to have a steady state of flow in the original physical domain. To avoid this, a special technique called "motionless viewing window" is applied as in the classical Lagrangian method. Accordingly, the column of cells which have moved out of the original physical domain to the right are deleted, while a new column of cells are added at the input flow boundary on the left.

In the Figs 10 to 12 we show computed Froude number distributions using our unified code for $h = 0, h = 0.5$ and $h = 0.999$, compared with the exact solutions. The poor resolution of the slip line seen in Fig.10 is a common feature of any method based on Eulerian coordinates, as a result of Godunov averaging across slip lines which, in general, do not coincide with the (Eulerian) coordinate lines. A comparison of Figs. 10 to 12 also shows that the slip line resolution improves with increasing h from $h = 0$ to $h = 0.999$, as expected.

Fig. 13 shows the computed Froude number using the grid-angle preserving h as determined by Eq.(57), which is solved at each time step using the method of characteristics. We see that its slip line resolution is almost as sharp as that for $h = 0.999$ and it is much better resolved than those for $h = 0$ and $h = 0.5$.

All the computations started with the Eulerian grid (Fig. 14). The flow-generated grids, i.e. the lines joining the cell centers, at steady state are shown in Figs 15 to 17. We note that: (a) the grid using grid angle preserving h is everywhere orthogonal, (b) the grids for $h = 0.5$ and $h = 0.999$ are severely deformed near the slip line, and such grid deformation, although doesn't bring any troubles in the present example, can cause inaccuracy in other steady flows [1] as well as for the unsteady flows; see example 2 above and the following examples.

Example 4. The next example is the "implosion/explosion" problem, so called by analogy with gas dynamics. It is an unsteady problem in a two-dimensional container. Initially, two regions of still water are separated by a cylindrical wall (radius 0.2) centered in the 1×1 square domain shown in Fig. 18. The depth of the water is 0.1 within the cylinder and 1 outside. At $t = 0$ the wall is removed and the resulting flow is investigated. Initially, a uniform rectangular grid 100×100 with $\Delta\xi = \Delta\eta = \Delta x = \Delta y = 0.01$ is given (Fig.19). We test this example with $h = 0$, $h = 0.5$, $h = 0.999$, and the grid-angle preserving h. When $h = 0.999$, the code can run only until $t = 0.04$; soon afterwards it breaks down. This is because the computational cells literally move with the fluid particles and for large h, become severely deformed. If we reduce h, say $h = 0.5$, the code can run longer until $t = 0.08$ when it, too, breaks down. This shows that smaller h can delay the severe cell deformation, but cannot remove it. With the grid-angle preserving method, which keeps the grid regular, the code can run for very long time without any indication of severe grid deformation. Figures 20-22 give the grids at $t = 0.04$ for different cases. We see that irregular grids prevail when h is constant and a regular grid prevails when the grid-angles are preserved (Fig. 22).

The surfaces of water height for different times are presented in the Fig.23. The physical behaviour of the flow is clearly captured in these pictures. After the wall is removed, the cylindrical shock moves inside the lower level region and the expansion wave propagates towards the walls (Fig.23a,b,c). At around time $t = 0.0565$ the shock collapses, resulting in a spectacular rise of water level near the center (Fig.23d). In subsequent times the water height decreases in the central region manifesting in cylindrical shock wave propagating now towards the walls (Fig.23e,f,g,h). This process of the water in the central region going up and down repeats itself continuingly, but with more moderate height and depth. We see in the Fig.23b,c,e that the column of water does not have perfect circular shape, as one would expect to have, but has wavy shape. It was shown in [22] that this behaviour is a consequence of using a rectangular grid.

Example 5. Finally, we test our code on the two-dimensional dam breaking problem. This test case was computed in, for example, [23,24].

Two levels of still water in the 1.4×1 basin are separated by a dam at the position $x = 0.7$ (Fig.24). The initial height ratio is 10, with values of $\zeta_l = 1$ and $\zeta_r = 0.1$ on two sides of an idealized dam that has been represented as a mathematical discontinuity in water. At time $t = 0$ water is released into the lower level side through a middle third of the dam, forming a wave that propagates while spreading laterally. At the same time, an expansion wave spreads into the reservoir.

A 140×100 rectangular grid was chosen in this case. We have performed the computation for $h = 0.999$ and for grid-angle preserving h. Figure 25 represents the flow-generated grid in the case $h = 0.999$ at time $t = 0.04$. The grid is severely distorted by the same reason as in the previous example, and, soon after that time the computation breaks down. The computation using grid-angle preserving h is stable, producing results for much larger time. In the Figure 26 we present a computed grid which is fairly uniform everywhere even at time $t = 0.15$. The surface elevation of the water height is given in Figure 27, describing the flow in details which are similar to those in [23,24].

6. Conclusions.

In this paper we have successfully adopted the uniform coordinates of Hui et.al. [1] for the shallow water equations. It has been tested on large number of problems and found that with the free function h in the unified coordinates chosen to preserve grid angles, the unified coordinate system is superior to both Eulerian and Lagrangian systems in that: **(a)** it resolves slip lines as sharply as the Lagrangian system, especially for steady flows, **(b)** it avoids the severe grid deformation of the Lagrangian system which causes inaccuracy and breakdown of computation, **(c)** it also automatically avoids spurious flow produced by the Lagrangian system. Additionally, it was also found that the two-dimensional shallow water equations in Lagrangian coordinates are only weakly hyperbolic, with possible defects in numerical computation.

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